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CONFIDENCE REGIONS FOR VARIANCE RATIOS IN VARIANCE COMPONENTS MODEL

by

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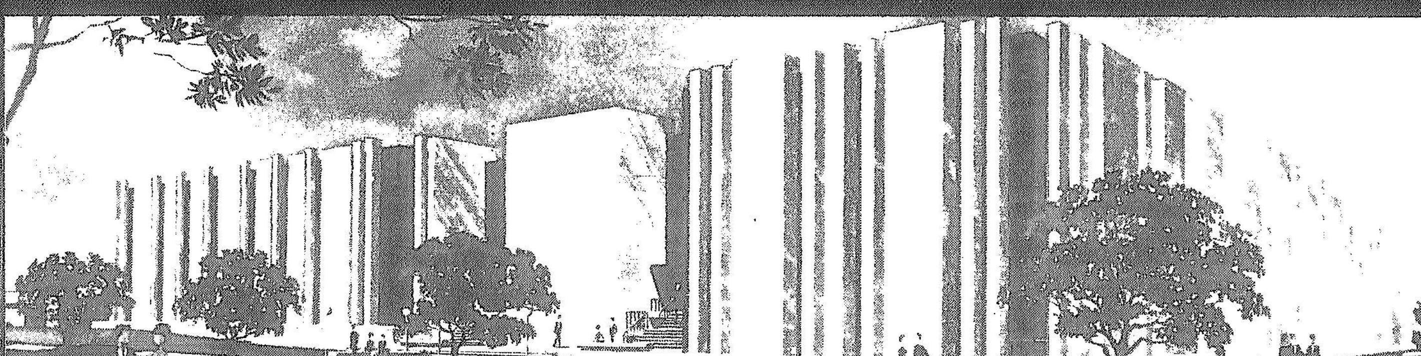

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CHAPTER I

INTRODUCTION

1.1 Description of the Problem

The situation frequently to be met in applied statistics is as follows: we have a set of data arranged in particular type of classification and described by a linear function of effects of various classes and subclasses. Generally this model is that which Eisenhart [4] has called Model II, in which all elements except μ are regarded as random variables, although it may frequently be called the Mixed Model, in which certain of the effects are regarded as fixed rather than random variables.

Mathematically, however, both Model II and the Mixed Model can be described by the general model

$$Y = X\alpha + U_1 b_1 + \ldots + U_c b_c + e, \tag{1.1}$$

where

Y is an nx1 observation vector;

X is an nxk matrix of known fixed numbers;

U_i is an nxm_i matrix of known fixed numbers;

α is an kx1 vector of unknown constants;

b_i is an m_ix1 vector of independent variables from $N(o, \sigma_i^2)$;

e is an nx1 vector of independent variables from $N(o, \sigma^2)$.

The citation on the following pages follow the style of the <u>Journal of the American Statistical Association</u>.

The unknown constants $\sigma_1^2, \ldots, \sigma_c^2$ and σ^2 are called the components of variance. Point estimates of variance components are now used in many fields of research. They seem to be more appropriate than interval estimates for many of the examples met in practice. However, a confidence interval is useful for assessing the accuracy of an estimate. If the confidence interval is wide, then little trust can be placed in a point estimate; if it is narrow, then the estimate can reasonably be regarded as trustworthy. Estimates do exist for the variances of the variance component estimates, but these being estimates, are less reliable than confidence intervals for assessing the accuracy of the variance component estimates. Also, they are less informative, since the usual type of variance component estimate has a complicated distribution, involving a nuisance parameter.

Most of the published papers on estimating confidence intervals for the components of variance and ratio of variances are concerned with the so-called balanced complete models, such as those with equal frequencies in the cells or subclasses of the one-way classification, nested classification and with factorial classifications. Usually the sums of squares of the analysis of variance is used to construct the confidence intervals. These procedures are not applicable to unbalanced models; since the analysis of variance sums of squares in this case are not, except for the error variance, distributed like a chi-square random variable.

In many instances, however, it is necessary to analyze data which are described by an unbalanced model. Although such data are quite common in applied statistics, it has received somewhat less attention than the balanced cases. This is due to the fact that the distribution of the ordinary sums of squares of the analysis of variance is, in general, complex. This fact hindered the development of a complete theory for the confidence region estimation of variance components (or variance ratios). Among the few papers on this subject is a paper by Hartley and Rao [10]. In this paper there is an outline of a procedure to derive a confidence region for the ratios of variances. Their procedure is as follows:

It is obvious that the vector of observations, Y, follows a multivariate normal distribution with covariance matrix

$$\sigma^2 H = \sigma^2[I + \gamma_1 U_1U_1' + \ldots + \gamma_c U_cU_c'], \tag{1.2}$$

where

$$\gamma_i = \sigma_i^2/\sigma^2. \tag{1.3}$$

The adjoined matrix

$$M = [X|U_1| \ldots |U_c] \tag{1.4}$$

is assumed to have as a base an nxr matrix W of the form

$$W = [X|U^*], \tag{1.5}$$

where U* has at leas one column from each U_i.

Let us write the model (1.1) in the form

$$Y = X\alpha + U^*\beta + Z, \tag{1.6}$$

where

β is a dummy null vector;

$Z = U_1b_1 + \ldots + U_cb_c + e.$

We now obtain the least squares estimates of α and β

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (W'H^{-1}W)^{-1} (W'H^{-1}Y). \tag{1.7}$$

The covariance matrix of $\hat{\alpha}$ and $\hat{\beta}$, $\Sigma(\gamma)$, can be partitioned into k and r-k sections as follows:

$$\Sigma(\gamma) = (W'H^{-1}W)^{-1} = \begin{bmatrix} \Sigma_{11}(\gamma) & \Sigma_{12}(\gamma) \\ \Sigma_{21}(\gamma) & \Sigma_{22}(\gamma) \end{bmatrix} . \tag{1.8}$$

Define

$$\Sigma_{22.1}(\gamma) = \Sigma_{22}(\gamma) - \Sigma_{21}(\gamma)\ \Sigma_{11}^{-1}(\gamma)\ \Sigma_{12}(\gamma); \tag{1.9}$$

$$\text{Reg} = (W'H^{-1}Y)'\ (W'H^{-1}W)^{-1}\ (W'H^{-1}Y); \tag{1.10}$$

$$\text{Res} = Y'H^{-1}Y - \text{Reg}. \tag{1.11}$$

Then the function

$$Z(\gamma) = \frac{(n-r)\ \hat{\beta}'\ \Sigma_{22.1}^{-1}(\gamma)\ \hat{\beta}}{(r-k)\ \text{Res}} \tag{1.12}$$

depends on the vector Y and the γ's. It is well known that the sampling distribution of the function (1.12) is an F distribution. Thus the inequality

$$Z(\gamma) \leq F(\alpha;\ r-k,\ n-r) \tag{1.13}$$

defines an exact confidence region for $\gamma_1, \ldots, \gamma_c$ if $F(\alpha;\ r-k,\ n-r)$ denotes the $100\alpha\%$ of F for r-k and n-r degrees of freedom.

1.2 Review of Literature

We have noted earlier that most of the published papers on estimating confidence intervals for the γ's are concerned with the balanced complete models. First we will review the methods available to construct confidence intervals for the γ's in the balanced model, then we will discuss Wald's method for constructing confidence intervals for the γ's.

Scheffe´ [16] discussed the one-way classification and showed that a confidence interval for γ_1 can be based on the classical F-ratio. This procedure is discussed in detail by Eisenhart et al. [5]. Graybill [7] showed how to test the hypothesis $\sigma_1^2 = 0$. This test is based also on the classical F-ratio; thus when translated to a confidence interval on γ_1 it becomes identical to that which Scheffe´ [16] discussed.

A more general problem was considered by Scheffe´ [15]. Let S_1 and S_2 be two independent random variates such that $n_i S_i/\sigma_i^2$ (i = 1, 2) is a central chi-square random variate with n_i degrees of freedom. The function $S_1/\theta S_2$, where $\theta = \sigma_1^2/\sigma_2^2$, is used to set confidence limits on θ. If S_1 is the Between Mean Squares and S_2 is the Within Mean Squares in the one-way classification, θ is equal to $1 + \gamma_1$. In general θ is a ratio of two nonhomogenous linear functions of $\gamma_1, \ldots, \gamma_c$. He showed that the logarithmically shortest confidence interval for θ exists and is unique although it is difficult to calculate if n_1 and n_2 are different.

Let us now suppose that $S_1, \ldots, S_k$ are independent random variables such that $n_i S_i/\sigma_i^2$ $(i = 1, \ldots, k)$ is a central chi-square random variate with n_i degrees of freedom. A simultaneous confidence statement on $\gamma_i = \sigma_i^2/\sigma_k^2$ $(i = 1, \ldots, k-1)$ based on the probability statement $P_r[F_{1,1} \le F_1 \le F_{1,2}, \ldots, F_{k-1,1} \le F_{k-1} \le F_{k-1,2}] = 1-\alpha$, where $F_i = S_i/\theta S_k$, was derived by Gnanadesikan [6]. When $S_1, \ldots, S_k$ are k mean squares of the analysis of variance of the balanced model, such a probability statement is known as "simultaneous analysis of variance". Computation of the numbers F_{i1} and F_{i2} $(i = 1, \ldots, k-1)$ requires tedious numerical integration. Broemeling [1] derived a conservative confidence region for $\theta_1, \ldots, \theta_{k-1}$ by choosing F_{i1} and F_{i2} to satisfy $Pr[F_{i1} \le F_i \le F_{i2}] = 1-\alpha_i$ and $1-\alpha = \Pi(1-\alpha_i)$. We note in passing that nothing is known about the optimal choice of F_{i1} and F_{i2}. The same approach was followed by Krishnaiah [13] to construct a simultaneous confidence region for the ratios $\sigma_i^2/\sigma_{i+1}^2$ $(i = 1, \ldots, k-1)$.

Using a randomization device, exact confidence limits were constructed by Heally [11]. The resulting confidence limits depend on the sums of squares of the analysis of variance table and on auxiliary observations on a random variable with known normal distribution. A consequence is that two statisticians confronted with the same analysis of variance table will in general construct different confidence limits.

Green [9] presented an approximate confidence interval for the expected value of the difference between two quantities which are independently distributed proportionally to chi-square. However, the solution is not presented in a form suitable for immediate practical application. Bross [2] gave a solution to this problem that can be easily computed, but Tukey [19] pointed out that the solution was wrong and proposed another solution. Huitson [12] has developed a procedure for the general problem of finding confidence limits for a linear combination of variances. Huitson was mainly concerned with the problem of estimating the total variability (that is, the sum of two or more variances). Bulmer [3] indicated that Huitson's expansion is not satisfactory in the case of the difference between two variances and developed an approximate solution for this case. Bulmer's solution is discussed in detail by Scheffé [16]. He pointed out that these confidence limits can be seriously invalidated by non-normality. Approximate confidence intervals for variance ratios specifying genetic heritability have been given by Graybill et al. [8]. This method is applicable to data specified by a balanced nested model. The heritability first is estimated by a ratio of a linear combination of chi-square variates to a chi-square variate; then its distribution is approximated by an F distribution. A probability statement is derived based on this distribution.

All exact and approximate procedures which we have discussed previously are valid only when the sums of squares of the analysis of variance are distributed proportionally to chi-square variates. For unbalanced data the sums of squares if formed by analogy to the balanced case do not satisfy this condition; consequently, the preceding procedures are not appropriate. In a series of papers Wald proposed a procedure for constructing a confidence region for the ratios of variances that is applicable to any balanced or unbalanced data. His first paper, Wald [20], dealt with the unbalanced one-way classification. He showed that a weighted Between Mean Squares is distributed proportionally to a chi-square variate. The ratio of this mean squares to the Within Mean Squares, which follows an F distribution is used to construct confidence limits on γ_1. Later this procedure was extended by Wald [21] to multiple classifications. A more general procedure, which includes the previous two procedures as special cases, was developed by Wald [22]. According to this procedure for each b_i a simple least squares estimates, $\hat{b}_i$, and its covariance matrix, V_i, are derived. It is to be noted here, that a simple least square means a least square assuming fixed effect model; the covariance matrix is computed assuming the elements b_i are random. The ratio of the quadratic form

$$q = (\hat{b}_i' V_i^{-1} \hat{b}_i)/(m_i-1) \tag{1.14}$$

to the Error Mean Squares is used to define a confidence region for the γ's. Define this ratio of quadratic forms by $Q(q, s^2)$, i.e.

$$Q(q, s^2) = q/s^2, \tag{1.15}$$

where

$$s^2 = \text{Error Mean Squares.}$$

Since Wald omitted the proof that $Q(q, s^2)$ follows an F distribution, we will give it here. $Q(q, s^2)$ follows an F distribution if the quadratic forms $(m_i-1)q$ and $N_e s^2$ are independent and each follows the chi-square distribution. The last condition is obviously satisfied. To prove the independence of $(m_i-1)q$ and $N_e s^2$ let $G(q, s^2|b)$ be the conditional density function of q and s^2 assuming $B' = [b_1', \ldots, b_c']$ to be fixed. Since $\hat{b}$ and s^2 are independent

$$G(q, s^2|b) = G_1(q|b)G_2(s^2|b), \tag{1.16}$$

where

$G_1(q|b)$ is the conditional distribution of q.

$G_2(s^2|b)$ is the conditional distribution of s^2.

But s^2 is independent of b thus

$$G_2(s^2|b) = H(s^2). \tag{1.17}$$

The unconditional joint density function of q and s^2 is

$$g(q, s^2) = \int f(b)\ G_1(q|b)\ H(s^2)db, \tag{1.18}$$

where

f(b) is the density of b.

But

$$g(q, s^2) = H(s^2) \int f(b)\ G_1(q|b)db. \tag{1.19}$$

Thus $g(q, s^2)$ is the product of density functions of s^2 and q which imply the independence of q and s^2.

Wald thought that V_i depends only on γ_i and derived a confidence limits for each γ_i separately. Unfortunately, as was pointed out by Spjøtvoll [17], the matrix V_i usually depends on $\gamma_1, \ldots, \gamma_c$. Wald's method, although it is quite general, is relatively unpopular. Thompson [18] used Wald's method to construct a confidence interval for γ_1 in a mixed partially balanced incomplete block design. His result is complicated unless the design is dual. Spjøtvoll [17] considered a Model II for a two-way classification with interaction and applied Wald's method to construct a confidence interval for the ratio of the interaction components to the error. He also outlined the procedure for constructing confidence intervals for the other two ratios of variances when the interaction is assumed zero.

CHAPTER II

BALANCED DESIGNS

In this chapter we shall discuss balanced designs. We shall construct confidence regions for the γ's in the one-way classification, nested classification, two-way classification and balanced incomplete block design.. We shall derive the confidence regions first using the method developed in Hartley and Rao's [10] paper, then we shall derive the regions using Wald's results [22].

We note here that if the fixed effects and the random effects are orthogonal the two procedures are identical if the following conditions are satisfied:

(1) The least squares estimator of β, the null vector, is identical to the simple least squares estimator.

(2) The Error Sums of Squares when all the factors of the model (1.1) are assumed fixed is identical to Res of Equation (1.11).

Note that when Error Sum of Squares is equal to Res then the two procedures will be identical if

$$\hat{\beta}' \Sigma_{22.1}^{-1}(\gamma)\hat{\beta} = \hat{b}' V^{-1}\hat{b}. \tag{2.1}$$

But under the assumption of orthogonality of the fixed and random effects

$$\Sigma_{22.1}(\gamma) = \Sigma_{22}(\gamma). \tag{2.2}$$

Now condition (1) above implies that

$$\mathrm{var}(\hat{\beta}) = \mathrm{var}(\hat{b}). \tag{2.3}$$

We know that $\quad \mathrm{var}(\hat{\beta}) = \Sigma_{22}(\gamma).$ $\quad$ (2.4)

Therefore

$$\hat{\beta}'\Sigma_{22.1}^{-1}(\gamma)\hat{\beta} = \hat{b}'\ V^{-1}\ \hat{b}. \tag{2.5}$$

2.1 One-way Classification

Let us suppose that we have r classes and s observations in each class. We can write the model as follows:

$$Y = 1\mu + Ub + e, \tag{2.6}$$

where

1 is an nx1 vector whose elements are unity;

$$U = \begin{bmatrix} A(1) \\ \vdots \\ A(r) \end{bmatrix}_{nxr}$$

A(k) is an sxr matrix whose elements, a_{ij}, are defined as follows:

$a_{ik} = 1$ for all i

$a_{ij} = 0$ for all i and $j \neq k$.

It is obvious that the covariance matrix of the observation vector is a block diagonal matrix

$$\sigma^2 H = \sigma^2\ \mathrm{Diag}\{(I + \gamma_1 J),\ \ldots,\ (I + \gamma_1 J)\}, \tag{2.7}$$

where J is a matrix whose elements are unity.

By direct multiplication it can be verified that

$$H^{-1} = \mathrm{Diag}\ \{(I - \frac{\gamma_1}{1 + s\gamma_1}\ J),\ \ldots,\ (I - \frac{\gamma_1}{1 + s\gamma_1}\ J)\}. \tag{2.8}$$

Following Hartley and Rao's [10] procedure which was discussed in chapter I we choose the matrix

$$W = [1|U^*], \tag{2.9}$$

where

$$U^* = \begin{bmatrix} A(1) \\ \vdots \\ A(r-1) \\ -J \end{bmatrix}_{nxr-1}, \tag{2.10}$$

J is an sxr-1 matrix whose elements are unity

A(k) is an sxr-1 matrix as defined previously,

as a base matrix. It can be easily verified that the normal equations for estimating μ and the dummy null vector β are

$$\frac{1}{1 + s\gamma_1} \begin{bmatrix} rs & 0 \\ 0 & s(I + J) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\beta} \end{bmatrix} = \frac{1}{1 + s\gamma_1} \begin{bmatrix} Y_{..} \\ Y_{1.} - Y_{r.} \\ \vdots \\ Y_{r-1} - Y_{r.} \end{bmatrix}, \tag{2.11}$$

where

$Y_{i.}$ is the total of the observations in the ith class,

$$Y_{..} = Y_{1.} + \ldots + Y_{r.}. \tag{2.12}$$

After simple algebraic manipulations the solution to the normal equations may be put in the following form

$$\begin{bmatrix} \hat{\mu} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} Y_{..}/rs \\ (Y_{1.} - Y_{..}/r)/s \\ \vdots \\ (Y_{r-1.} - Y_{..}/r)/s \end{bmatrix}. \tag{2.13}$$

The Reg and Res quantities are

$$\text{Reg} = [1/(1 + s\gamma_1)]\ [Y_{..}^2/rs + (\Sigma Y_{i.}^2 - Y_{..}^2/r)/s], \tag{2.14}$$

$$\text{Res} = Y'Y - (\Sigma Y_{i.}^2)/s. \tag{2.15}$$

The covariance matrix of the estimates in equation (2.13) is

$$\Sigma(\gamma) = \begin{bmatrix} 1/rs & 0 \\ 0 & (1/s)(I - 1/rJ) \end{bmatrix}, \tag{2.16}$$

and thus

$$\Sigma_{22.1}^{-1}(\gamma) = [s/(1 + s\gamma_1)][I + J]. \tag{2.17}$$

Substituting in equation (1.12) it can be easily verified that

$$Z(\gamma_1) = \frac{r(s - 1)\ \Sigma(Y_{i.} - Y_{..}/r)^2/(r - 1)}{s(1 + s\gamma_1)[Y'Y - \Sigma Y_{i.}^2/s]} \ . \tag{2.18}$$

After simplifications the confidence interval for γ_1 becomes

$$1/s \left[\frac{r(s - 1)\ \Sigma(Y_{i.} - Y_{..}/r)^2}{s(Y'Y - \Sigma Y_{i.}/s)F_2\ (r - 1)} -1 \right] \leq \gamma_1$$

$$\leq 1/s \left[\frac{r(s - 1)\ \ (Y_{i.} - Y_{..}/r)^2}{s(Y'Y - \Sigma Y_{i.}^2/s)F_1(r - 1)} - 1 \right]. \tag{2.19}$$

This confidence interval is identical to the classical analysis of variance interval discussed in Scheffe´ [16]. Equation (2.16) shows that the fixed effect, in this case the mean, and the random effects are orthogonal. Thus Wald's procedure will yield the same confidence interval since the estimate $\hat{\beta}$ in equation (2.13) is identical to the simple least squares estimate and the quantity Res of equation (2.15) is identical to the Error Sum of Squares.

2.2 Nested Classification

The balanced twofold nested classification components of variance model is defined by the model

$$Y = 1\mu + U_1 b_1 + U_2 b_2 + e, \tag{2.20}$$

where

1 is an nxl vector whose elements are unity,

$$U_1 = \begin{bmatrix} A(1) \\ \vdots \\ \underline{A}(r) \end{bmatrix}_{rstxr}, \tag{2.21}$$

A(k) is as defined previously,

$$U_2 = \text{Diag } \{B, \ldots, B\}, \tag{2.22}$$

$$B = \begin{bmatrix} A(1) \\ \vdots \\ A(s) \end{bmatrix}_{stxs}. \tag{2.23}$$

The covariance matrix of the vector of observations is

$$\sigma^2 H = \sigma^2[I + \gamma_1 U_1 U_1' + \gamma_2 U_2 U_2'], \tag{2.24}$$

where

$$U_1 U_1' = \text{Diag}\{J_{st}, \ldots, J_{st}\}, \tag{2.25}$$

$$U_2 U_2' = \text{Diag}\{J_t, \ldots, J_t\}. \tag{2.26}$$

By direct multiplication we can verify that

$$H^{-1} = I + \theta_1 U_1 U_1' + \theta_2 U_2 U_2', \tag{2.27}$$

where

$$\theta_1 = \frac{-\gamma_1}{(1 + t\gamma_2)(1 + t\gamma_2 + st\gamma_1)}, \tag{2.28}$$

$$\theta_2 = \frac{-\gamma_2}{1 + t\gamma_2}. \tag{2.29}$$

Following the procedure described in Hartley and Rao [10] we choose the base matrix

$$W = [1|U_1^*|U_2^*], \tag{2.30}$$

where

$$U_1^* = \begin{bmatrix} A(1) \\ \vdots \\ A(r-1) \\ -J \end{bmatrix}_{rst\times(r-1)} \tag{2.31}$$

$$U^*_2 = \mathrm{Diag}\{B^*, \ldots, B^*\}, \tag{2.32}$$

$$B^* = \begin{bmatrix} A(1) \\ \vdots \\ A(s-1) \\ -J \end{bmatrix}_{st\times(s-1)}, \tag{2.33}$$

and the submatrices in equations (2.31), (2.33) are defined earlier. It is convenient to divide the dummy null vector β into r-1 and r(s-1) subvectors as follows

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}. \tag{2.34}$$

The normal equations for the least squares estimation of μ and β are given by (2.35). After a simple algebraic manipulation the solution of the normal equation is given by (2.40). It is easy to verify that the quantities Reg and Res are given by equation (2.41) and (2.42) respectively.

$$\begin{bmatrix} \delta rst & 0 & 0 \\ 0 & \delta st(I+J) & 0 \\ 0 & 0 & t(1+t\theta_2)\,C \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \delta Y_{...} \\ \delta(Y_{1..} - Y_{r..}) \\ \vdots \\ \delta(Y_{r-1..} - Y_{r..}) \\ (1+t\theta_2)(Y_{11.} - Y_{1s.}) \\ \vdots \\ (1+t\theta_2)(Y_{1s-1.} - Y_{1s.}) \\ \vdots \\ (1+t\theta_2)(Y_{r1.} - Y_{rs.}) \\ \vdots \\ (1+t\theta_2)(Y_{rs-1.} - Y_{rs.}) \end{bmatrix} \tag{2.35}$$

where

$$\delta = (1 + t\gamma_2 + st\gamma_1)^{-1}, \tag{2.36}$$

$$C = \text{Diag}\{(I + J), \ldots, (I + J)\}, \tag{2.37}$$

$Y_{ij.}$ is the total of observations in the ijth subclass,

$$Y_{i..} = Y_{i1.} = \ldots + Y_{is.}, \tag{2.38}$$

$$Y_{...} = Y_{1..} + \ldots + Y_{r..} \ . \tag{2.39}$$

$$\begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} Y_{...}/rst \\ (Y_{1..} - Y_{...}/r)/st \\ \vdots \\ (Y_{r-1..} - Y_{..}/r)/st \\ (Y_{11.} - Y_{1..}/s)/t \\ \vdots \\ (Y_{1s-1.} - Y_{1..}/s)/t \\ \vdots \\ (Y_{r1.} - Y_{r..}/s)/t \\ \vdots \\ (Y_{rs-1.} \ Y_{r..}/s)/t \end{bmatrix} \tag{2.40}$$

$$\text{Reg} = \frac{Y^2_{\cdot\cdot\cdot}}{rst(1 + t\gamma_2 + st\gamma_1)} + \frac{1}{st(1 + t\gamma_2 + st\gamma_1)} [\Sigma Y^2_{i\cdot\cdot} - Y^2_{\cdot\cdot\cdot}/r] +$$

$$\frac{1}{t(1 + t\gamma_2)} [\Sigma Y^2_{ij\cdot} - \Sigma Y^2_{j\cdot\cdot}/s]. \tag{2.41}$$

$$\text{Res} = Y'Y - \Sigma Y^2_{ij\cdot}/t. \tag{2.42}$$

It can be easily seen that the covariance matrix of the estimates in equation (2.40) is

$$\Sigma\ (\gamma) = \begin{bmatrix} \dfrac{1 + t\gamma_2 + st\gamma_1}{rst} & 0 & 0 \\ 0 & \Sigma_{22}(\gamma) & 0 \\ 0 & 0 & \Sigma_{33}(\gamma) \end{bmatrix}, \tag{2.43}$$

where

$$\Sigma_{22}\ (\gamma) = \frac{1 + t\gamma_2 + st\gamma_1}{st}\ (I - 1/rJ), \tag{2.44}$$

$$\Sigma_{33}\ (\gamma) = \frac{1 + t\gamma_2}{t} \text{Diag}\{(I - 1/sJ), \ldots, (I - 1/sJ)\}. \tag{2.45}$$

The mutual orthogonality of $\hat{\mu}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ enables us to derive several confidence regions. These confidence regions are identical to the regions derived from the analysis of variance table. Wald's method gives identical confidence regions. This is obvious since the estimates $\hat{\mu}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are orthogonal to each other and identical to the simple least squares estimates and the quantity Res defined by equation (2.42) is identical to the Error Sum of Squares in the analysis of variance. Let us now give the confidence regions.

(1) A confidence interval for γ_2 may be based on the function

$$Z_1(\gamma_2) = \frac{rs(t - 1)\ \hat{\beta}_2'\ \Sigma_{33}^{-1}\ (\gamma)\ \hat{\beta}_2}{r(s - 1)\ \text{Res}}\ . \tag{2.46}$$

It is readily seen that

$$Z_1(\gamma_2) = \frac{rs(t - 1)[\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s]}{t(1 + t\gamma_2)\ r(s - 1)\ \text{Res}}\ , \tag{2.47}$$

and the confidence interval for γ_2 is

$$1/t \left[\frac{rs(t - 1)[\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s]}{rt(s - 1)\ \text{Res}\ F_2} - 1\right] \leq \gamma_2 \leq$$

$$1/t \left[\frac{rs(t - 1)[\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2 .s]}{rt(s - 1)\ \text{Res}\ F_1} - 1\right], \tag{2.48}$$

(2) A confidence region for γ_1 and γ_2 can be derived from the function

$$Z_2(\gamma_1,\ \gamma_2) = \frac{rs(t - 1)\ \hat{\beta}_1'\ \Sigma_{22}^{-1}(\gamma)\ \hat{\beta}_1}{(r - 1)\ \text{Res}}\ . \tag{2.49}$$

After simple algebraic manipulations we obtain

$$Z_2(\gamma_1,\ \gamma_2) = \frac{rs(t - 1)[\Sigma Y_{i..}^2 - Y_{...}^2/r]}{(r - 1)st(1 + t\gamma_2 + st\gamma_1)\ \text{Res}}\ , \tag{2.50}$$

and

$$\frac{rs(t - 1)[\Sigma Y_{i..}^2 - Y_{...}^2/r]}{st(r - 1)\ \text{Res}\ F_2} \leq 1 + t\gamma_2 + st\gamma_1 \leq$$

$$\frac{rs(t - 1)[\Sigma Y_{i..}^2 - Y_{...}^2/r]}{st(r - 1)\ \text{Res}\ F_1}\ . \tag{2.51}$$

(3) Another confidence region for γ_1 and γ_2 can be based on the function

$$Z_3(\gamma_1, \gamma_2) = \frac{rs(t-1)[\hat{\beta}_1' \Sigma_{22}^{-1}(\gamma) \hat{\beta}_1 + \hat{\beta}_2' \Sigma_{33}^{-1} \hat{\beta}_2]}{(rs-1) \text{ Res}} . \tag{2.52}$$

Simplifying (2.53) we get

$$Z_3(\gamma_1, \gamma_2) = \frac{rs(t-1)\left[\dfrac{\Sigma Y_{i..}^2 - Y_{...}^2/r}{st(1+t\gamma_2+st\gamma_1)} + \dfrac{\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s}{t(1+t\gamma_2)}\right]}{(rs-1) \text{ Res}} . \tag{2.51}$$

The confidence region is defined by the inequality

$$\frac{(rs-1) \text{ Res } F_1}{rs(t-1)} \le \frac{\Sigma Y_{i..}^2 - Y_{...}^2/r}{st(1+t\gamma_2+st\gamma_1)} + \frac{\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s}{t(1+t\gamma_2)} \le$$

$$\frac{(rs-1) \text{ Res } F_2}{rs(t-1)} . \tag{2.54}$$

(4) Finally a confidence region for γ_1 and γ_2 can be derived from

$$Z_4(\gamma_1, \gamma_2) = \frac{r(s-1)\ \hat{\beta}_1' \Sigma_{22}^{-1}(\gamma)\ \hat{\beta}_1}{(r-1)\ \hat{\beta}_2' \Sigma_{33}^{-1}(\gamma)\ \hat{\beta}_2} , \tag{2.55}$$

or from its inverse

$$Z_5(\gamma_1, \gamma_2) = \frac{1}{Z_4(\gamma_1, \gamma_2)} \tag{2.56}$$

After simplifications we see that the confidence region derived from (2.55) is defined by the inequality

$$\frac{st(r-1)[\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s]\ F_1}{tr(s-1)[\Sigma Y_{i..}^2 - Y_{...}^2/r]} \le \frac{1+t\gamma_2}{1+t\gamma_2+st\gamma_1} \le$$

$$\frac{st(r-1)[\Sigma Y_{ij.}^2 - \Sigma Y_{i..}^2/s]\ F_2}{tr(s-1)[\Sigma Y_{i..}^2 - Y_{...}^2/r]} \tag{2.57}$$

2.3 Two-way Classification Mixed Model without Interaction

The two-way classification Mixed Model without interaction is specified by the mathematical model

$$Y = 1\mu^* + T\tau + Ub + e, \tag{2.58}$$

where

1 is an nxl vector whose elements are unity,

$$T = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}_{rstxs}, \tag{2.59}$$

$$B = \begin{bmatrix} A(1) \\ \vdots \\ A(s) \end{bmatrix}_{stxs}, \tag{2.60}$$

$$U = \begin{bmatrix} A(1) \\ \vdots \\ A(r) \end{bmatrix}_{rstxr}. \tag{2.61}$$

The matrices A(k) are defined previously. Let us note here that τ is a vector of fixed elements and b is a vector of random elements. It is well known that the matrix T is not of full column rank; thus in order to apply the procedure described by Hartley and Rao [10] we need to "reparameterize" the model (2.58). This might be achieved by redefining the model as follows

$$Y = 1\mu + X\alpha + Ub + e, \tag{2.62}$$

where

$$\mu = \mu^* + \Sigma\tau_i/s, \tag{2.63}$$

α is an (s - 1)xl vector whose elements α_i are defined as follows:

$$\alpha_i = \tau_i - \Sigma\tau_i/s, \tag{2.64}$$

$$X = \begin{bmatrix} B^* \\ \vdots \\ B^* \end{bmatrix}_{rst \times (s-1)}, \tag{2.65}$$

$$B^* = \begin{bmatrix} A(1) \\ \vdots \\ A(s-1) \\ -J \end{bmatrix}_{st \times (s-1)}, \tag{2.66}$$

Now we can apply the procedure to the model (2.62).

The covariance matrix of the observations is

$$\sigma^2 H = \sigma^2 \,\mathrm{Diag}\{(I + \gamma_1 J), \ldots, (I + \gamma_1 J)\}, \tag{2.67}$$

and

$$H^{-1} = \mathrm{Diag}\{(I - \frac{\gamma_1}{1 + st\gamma_1} J), \ldots, (I - \frac{\gamma_1}{1 + st\gamma_1} J)\}. \tag{2.68}$$

Let now choose W as a base matrix, where

$$W = [1|X|U^*], \tag{2.69}$$

$$U^* = \begin{bmatrix} A(1) \\ \vdots \\ A(r-1) \\ -J \end{bmatrix}_{rst \times (r-1)} . \tag{2.70}$$

It can be easily verified that the normal equations for the least squares estimates of μ, α and the dummy null vector β are

$$\begin{bmatrix} \frac{rst}{1 + st\gamma_1} & 0 & 0 & 0 \\ 0 & rt(I + J) & & 0 \\ 0 & 0 & \frac{st}{1 + st\gamma_1} & (I + J) \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} Y_{...}/(1 + st\gamma_1) \\ Y_{.1.} - Y_{.s.} \\ \vdots \\ Y_{.s-1.} - Y_{.s.} \\ \frac{Y_{1..} - Y_{r..}}{1 + st\gamma_1} \\ \vdots \\ \frac{Y_{r-1..} - Y_{r..}}{1 + st\gamma_1} \end{bmatrix},$$

(2.71)

where

$Y_{.j.}$ is the total of all the observations that received the jth fixed factor,

$Y_{i..}$ is the total of all the observations that received the ith random factor,

$Y... = Y_{1..} + ... + Y_{r..}$.

Solving (2.71) we obtain

$$\begin{bmatrix} \hat{\mu} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} Y.../rst \\ (Y_{.1.} - Y.../s)/rt \\ \vdots \\ (Y_{.s-1.} - Y.../s)/rt \\ (Y_{1..} - Y.../r)/st \\ \vdots \\ (Y_{r-1..} - Y.../r)/st \end{bmatrix} . \tag{2.72}$$

It readily seen that

$$\text{Reg} = \frac{Y^2...}{rst(1 + st\gamma_1)} + \frac{1}{rt} [\Sigma Y^2_{.j.} - Y^2.../s]$$

$$+ \frac{1}{st(1 + st\gamma_1)} [\Sigma Y^2_{j..} - Y_{...}/r], \tag{2.73}$$

$$\text{Res} = Y'Y - \Sigma Y^2_{i..}/st - \Sigma Y^2_{.j.}/rt + Y^2.../rst, \tag{2.74}$$

$$\Sigma^{-1}_{22.}(\gamma) = \frac{st}{1 + st\gamma_1} (I + J). \tag{2.75}$$

The confidence interval is based on the function

$$Z(\gamma_1) = \frac{(rst - r - s + 1)\ \hat{\beta}'\ \Sigma^{-1}_{22.1}(\gamma)\ \hat{\beta}}{(r - 1)\ \text{Res}} . \tag{2.76}$$

After simple algebraic manipulations we get

$$\frac{1}{st}\left[\frac{(rst - r - s + 1)(\Sigma Y_{i..}^2 - Y_{...}^2/r)}{st(r - 1)\ \text{Res}\ F_2} - 1\right] \le \gamma_1$$

$$\le \frac{1}{st}\left[\frac{(rst - r - s + 1)(\Sigma Y_{i..}^2 - Y_{...}^2/r)}{st(r - 1)\ \text{Res}\ F_1} - 1\right] . \qquad (2.77)$$

It can be readily verified that (2.76) yield the classical F ratio. Thus the inequality (2.72) can be obtained from the analysis of variance table. We note that the fixed elements and the random elements of the model (2.62) are orthogonal, furthermore conditions (1) and (2) on page 11 are satisfied. Thus Wald's procedure will yield the same result as in (2.77).

2.4 Two-way Classification Model II without Interaction

The model of the two-way classification Model II without interaction may be written in the following form

$$Y = 1\mu + U_1 b_1 + U_2 b_2 + e, \qquad (2.78)$$

where

$$U_1 = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}_{rst\times s}, \qquad (2.79)$$

$$B = \begin{bmatrix} A(1) \\ \vdots \\ A(s) \end{bmatrix}_{st\times s}, \qquad (2.80)$$

and

$$U_2 = \begin{bmatrix} A(1) \\ \vdots \\ A(r) \end{bmatrix}_{rstxr} . \tag{2.81}$$

The matrices A(k) were defined previously.

To find the covariance matrix of the observations, it is convenient to define

$$D_{stxst} = \mathrm{Diag}\{J_{txt}, \ldots, J_{txt}\}, \tag{2.82}$$

$$J(I) = \begin{bmatrix} I \cdots I \\ \cdots\cdots \\ I \cdots I \end{bmatrix}_{rstxrst}, \tag{2.83}$$

where I is the identity matrix of dimension st. Now it is readily seen that the covariance matrix is

$$\sigma^2 H = \sigma^2[I + \gamma_1 \,\mathrm{Diag}\{D, \ldots, D\}J(I) + \gamma_2 \,\mathrm{Diag}\{J, \ldots, J\}]. \tag{2.84}$$

To find H^{-1} let us write H in the following form

$$H = \mathrm{Diag}\{(I + \gamma_2 J), \ldots, (I + \gamma_2 J)\}[I + \gamma_1 \mathrm{Diag}\{(D - \frac{t\gamma_2}{1 + st\gamma_2} J), \ldots, (D - \frac{t\gamma_2}{1 + st\gamma_2} J)\} J(I)]. \tag{2.85}$$

By direct multiplication it is easy to verify that

$$[I + \gamma_1 \mathrm{Diag}\{(D - \frac{t\gamma_2}{1 + st\gamma_2} J), \ldots, (D - \frac{t\gamma_2}{1 + st\gamma_2} J)\}J(I)]^{-1}$$

is equal to

$$[I - \frac{\gamma_1}{1 + rt\gamma_1} \mathrm{Diag}\{(D - \frac{t\gamma_2}{1 + rt\gamma_1 + st\gamma_2} J), \ldots, (D - \frac{t\gamma_2}{1 + rt\gamma_1 + st\gamma_2} J)\}J(I)] \tag{2.86}$$

Thus it is readily verified that

$$H^{-1} = [I - \frac{\gamma_1}{1 + rt\gamma_1} \ \mathrm{Diag}\{(D - \frac{t\gamma_2}{1 + rt\gamma_1 + st\gamma_2} J), \ldots,$$

$$(D - \frac{t\gamma_2}{1 + rt\gamma_1 + st\gamma_2} J)\} J(I)] \ \mathrm{Diag}\{(I - \frac{\gamma_2}{1 + st\gamma_2} J), \ldots,$$

$$(I - \frac{\gamma_2}{1 + st\gamma_2} J)\}. \tag{2.87}$$

Following the procedure described by Hartley and Rao [10] we choose

$$W = [1|U_1^*|U_2^*], \tag{2.88}$$

where

$$U_1^* = \begin{bmatrix} B^* \\ \vdots \\ B^* \end{bmatrix}_{rst \times (s-1)}, \tag{2.89}$$

$$B^* = \begin{bmatrix} A(1) \\ \vdots \\ A(s-1) \\ -J \end{bmatrix}_{st \times (s-1)}, \tag{2.90}$$

$$U_2^* = \begin{bmatrix} A(1) \\ \vdots \\ A(r-1) \\ -J \end{bmatrix}_{rst \times (r-1)}, \tag{2.91}$$

as a base matrix.

It is convenient to divide the dummy null vector β into two subvectors of dimension $s - 1$ and $r - 1$ as follows

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}. \tag{2.92}$$

Now it can be easily verified that the normal equations for the estimation of μ and the dummy null vector β are

$$\begin{bmatrix} \frac{rst}{1+rt\gamma_1+st\gamma_2} & 0 & 0 \\ 0 & \frac{rt}{1+rt\gamma_1}[I+J] & 0 \\ 0 & 0 & \frac{st}{1+st\gamma_2}[I+J] \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \frac{Y_{...}}{1+rt\gamma_1+st\gamma_2} \\ \frac{Y_{.1.}-Y_{.s.}}{1+rt\gamma_1} \\ \vdots \\ \frac{Y_{.s-1.}-Y_{.s.}}{1+rt\gamma_1} \\ \frac{Y_{1..}-Y_{r..}}{1+st\gamma_2} \\ \vdots \\ \frac{Y_{r-1..}-Y_{r..}}{1+st\gamma_2} \end{bmatrix}, \tag{2.93}$$

where

$Y_{.j.}$ is the total of observations received the jth random factor of b_1

$Y_{i..}$ is the total of observations received the ith random factor of b_2

$$Y_{...} = Y_{1..} + \dots + Y_{r..} \;. \tag{2.94}$$

Solving the normal equations (2.93) and simplifying we obtain

$$\begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} Y_{...}/rst \\ (Y_{.1.} - Y_{...}/s)/rt \\ \vdots \\ (Y_{.s-1.} - Y_{...}/s)/rt \\ (Y_{1..} - Y_{...}/r)/st \\ \vdots \\ (Y_{r-1..} - Y_{...}/r)/st \end{bmatrix} \tag{2.95}$$

$$\text{Reg} = \frac{Y^2_{\cdots}}{rst(1 + rt\gamma_1 + st\gamma_2)} + \frac{1}{rt(1 + rt\gamma_1)}[\Sigma Y^2_{.j.} - Y^2_{\cdots}/s]$$

$$+ \frac{1}{st(1 + st\gamma_2)}[\Sigma Y^2_{i..} - Y^2_{\cdots}/r], \tag{2.96}$$

$$Y'H^{-1}Y = Y'Y - [\frac{\gamma_2}{1 + st\gamma_2} - \frac{t\gamma_1^2(1 - \frac{1 + rt\gamma_1}{1 + st\gamma_2})}{(1 + rt\gamma_1)(1 + rt\gamma_1 + st\gamma_2)}]\Sigma Y^2_{i..}$$

$$- \frac{1}{1 + rt\gamma_1}[\Sigma Y^2_{ij.} + \Sigma_{i\neq j}\Sigma_k Y_{ik.} Y_{jk.}]$$

$$+ \frac{t\gamma_1^2(1 - \frac{1 + rt\gamma_1}{1 + st\gamma_2})}{(1 + rt\gamma_1)(1 + rt\gamma_1 + st\gamma_2)}\Sigma_{i\neq j} Y_{i..} Y_{j..} \,. \tag{2.97}$$

$$\text{Res} = Y'H^{-1}Y - \text{Reg}. \tag{2.98}$$

From (2.88) it is readily seen that the covariance matrix of the estimates (2.95) is

$$\Sigma(\gamma) = \begin{bmatrix} \frac{1 + rt\gamma_1 + st\gamma_2}{rst} & 0 & 0 \\ 0 & \Sigma_{22}(\gamma_1) & 0 \\ & 0 & \Sigma_{33}(\gamma_2) \end{bmatrix}, \tag{2.99}$$

where

$$\Sigma_{22}(\gamma_1) = \frac{1 + rt\gamma_1}{rt}[I - 1/sJ], \tag{2.100}$$

$$\Sigma_{33}(\gamma_2) = \frac{1 + st\gamma_1}{st}[I - 1/rJ], \tag{2.101}$$

It is possible to derive several confidence regions for γ_1 and γ_2.

(1) A confidence region for γ_1 and γ_2 can be based on the function

$$Z_1(\gamma_1,\gamma_2) = \frac{(rst - r - s + 1)[\hat{\beta}_1'\Sigma_{22}^{-1}(\gamma_1)\hat{\beta}_1 + \hat{\beta}_2'\Sigma_{33}^{-1}(\gamma_2)\hat{\beta}_2]}{(r + s - 2)\text{ Res}}. \tag{2.102}$$

This function can be simplified to

$$Z_1(\gamma_1,\gamma_2) = \frac{\frac{rst - r - s + 1}{rt(1 + rt\gamma_1)}[\Sigma Y_{.j.}^2 - Y_{...}^2/s]}{(r + s - 2)\text{ Res}} + \frac{\frac{rst - r - s + 1}{st(1 + st\gamma_2)}[\Sigma Y_{i..}^2 - Y_{...}^2/r]}{(r + s - 2)\text{ Res}}. \tag{2.103}$$

(2) It can be easily seen that the functions $Z_2(\gamma_1,\gamma_2)$ and $Z_3(\gamma_1,\gamma_2)$ given by (2.104) and (2.105) follows F distribution and thus a confidence regions may be based on them.

$$Z_2(\gamma_1,\gamma_2) = \frac{(rst - r - s + 1)\ \hat{\beta}_1'\Sigma_{22}^{-1}\ (\gamma_1)\hat{\beta}_1}{(s - 1)\text{ Res}}. \tag{2.104}$$

$$Z_3(\gamma_1,\gamma_2) = \frac{(rst - r - s + 1)\ \hat{\beta}_2'\Sigma_{33}^{-1}(\gamma_2)\hat{\beta}_2}{(r - 1)\text{ Res}}. \tag{2.105}$$

It is easy to verify that $Z_2(\gamma_1,\gamma_2)$ may be simplified to (2.106) and $Z_3(\gamma_1,\gamma_2)$ to (2.107)

$$Z_2(\gamma_1,\gamma_2) = \frac{\frac{rst - r - s + 1}{rt(1 + rt\gamma_1)}[\Sigma Y_{.j.}^2 - Y_{...}^2/s]}{(s - 1)\text{ Res}}. \tag{2.106}$$

$$Z_3(\gamma_1,\gamma_2) = \frac{\frac{rst - s - r + 1}{st(1 + st\gamma_2)}[\Sigma Y_{i..}^2 - Y_{...}^2/r]}{(r - 1)\text{ Res}}. \tag{2.107}$$

(3) It is readily verified that the quadratic forms

$$q_1(\gamma_1) = \hat{\beta}_1'\ \Sigma_{22}^{-1}(\gamma_1)\hat{\beta}_1 \tag{2.108}$$

and

$$q_2(\gamma_2) = \hat{\beta}_2' \ \Sigma_{33}^{-1}(\gamma_2)\hat{\beta}_2 \tag{2.109}$$

are independent and each is distributed as a chi-square with (s - 1) and (r - 1) degrees of freedom respectively. Thus the function

$$Z_4(\gamma_1,\gamma_2) = \frac{q_1(\gamma_1)/(s - 1)}{q_2(\gamma_2)/(r - 1)} \tag{2.110}$$

or its inverse

$$Z_5(\gamma_1,\gamma_2) = [Z_4(\gamma_1,\gamma_2)]^{-1} \tag{2.111}$$

follows F distribution. Either $Z_4(\gamma_1,\gamma_2)$ or $Z_5(\gamma_1,\gamma_2)$ can be used to derive a confidence region for γ_1 and γ_2. We derive the confidence region based on $Z_4(\gamma_1,\gamma_2)$. $Z_4(\gamma_1,\gamma_2)$ may be simplified to the form

$$Z_4(\gamma_1,\gamma_2) = \frac{(r - 1)st(1 + st\gamma_2)(\Sigma Y^2_{.j.} - Y^2_{...}/s)}{(s - 1)rt(1 + rt\gamma_1)(\Sigma Y^2_{i..} - Y^2_{...}/r)}. \tag{2.112}$$

Thus the confidence region is described by the inequality

$$F_1 \frac{(s - 1)r(\Sigma Y^2_{i..} - Y^2_{...}/r)}{(r - 1)s(\Sigma Y^2_{.j.} - Y^2_{...}/s)} \leq \frac{1 + st\gamma_2}{1 + rt\gamma_1} \leq F_2 \frac{(s - 1)r(\Sigma Y^2_{i..} - Y^2_{...}/r)}{(r - 1)s(\Sigma Y^2_{.j.} - Y^2_{...}/s)} \tag{2.113}$$

It is to be noted that the least squares estimates of μ and β are identical to the simple least squares estimates, but the quantity Res defined by equation (2.98) is different from the Error Sum of Squares in the analysis of variance assuming fixed effect model. Thus Wald's procedure [22] yields a different confidence regions. Let us derive Wald's confidence regions.

As in Hartley and Rao's [10] procedure, Wald's procedure provides several confidence regions, some of them are intervals for a single γ. We denote by ESSq the Error Sum of Squares assuming fixed model. Since the simple least squares estimates of μ and β are identical to least squares estimate, we know that

$$\mathrm{var}(\hat{\beta}_1) = \Sigma_{22}(\gamma_1),$$

and

$$\mathrm{var}(\hat{\beta}_2) = \Sigma_{33}(\gamma_1), \tag{2.115}$$

where $\Sigma_{22}(\gamma_1)$ and $\Sigma_{33}(\gamma_1)$ were defined by equations (2.100) and (2.101), respectively. Thus confidence regions, or intervals, derived from Wald's procedure may be based on the following functions.

$$Z_1(\gamma_1) = \frac{(rst - r - s + 1)\ q_1}{(s - 1)\ \mathrm{ESSq}}, \tag{2.116}$$

$$Z_2(\gamma_2) = \frac{(rst - r - s + 1)\ q_2}{(r - 1)\ \mathrm{ESSq}}, \tag{2.117}$$

$$Z_3(\gamma_1,\gamma_2) = \frac{(rst - r - s + 1)(q_1 + q_2)}{(r + s - 2)\ \mathrm{ESSq}}, \tag{2.118}$$

where q_1 and q_2 are defined by equations (2.108) and (2.109), respectively.

It is readily seen that $Z_1(\gamma_1)$, $Z_2(\gamma_2)$ and $Z_3(\gamma_1,\gamma_2)$ can be simplified to

$$Z_1(\gamma_1) = \frac{(rst - r - s + 1)(\Sigma Y^2_{.j.} - Y^2_{...}/s)}{rt(r - 1)(1 + rt\gamma_1)\ \mathrm{ESSq}}, \tag{2.119}$$

$$Z_2(\gamma_2) = \frac{(rst - r - s + 1)(\Sigma Y^2_{i..} - Y^2_{...}/r)}{st(s - 1)(1 + st\gamma_1)\ \mathrm{ESSq}}, \tag{2.120}$$

$$Z_3(\gamma_1,\gamma_2) = \frac{(rst - r - s + 1)\left[\frac{\Sigma Y^2_{.j.} - Y^2_{...}}{rt(1 + rt\gamma_1)} + \frac{\Sigma Y^2_{i..} - Y^2_{...}/r}{st(1 + st\gamma_1)}\right]}{(r + s - 2)\ ESSq}. \tag{2.121}$$

It is clearly seen that $Z_1(\gamma_1)$, $Z_2(\gamma_2)$ and $Z_3(\gamma_1,\gamma_2)$ may be derived from the analysis of variance table; thus Wald's procedure and the analysis of variance procedure are identical.

2.5 Balanced Incomplete Block Design

In this section we assume that the data are described by a balanced incomplete block design with block considered random.

It is convenient to start with general two-way classification Mixed Model with unequal numbers per subclass and without interaction. The model of such design is

$$Y = 1\mu + T\tau + Ub + e, \tag{2.122}$$

where

$$T = \begin{bmatrix} B_1 \\ \vdots \\ B_r \end{bmatrix}_{n_{..} \times s}, \tag{2.123}$$

$$B_i = \begin{bmatrix} A(1)n_{i1} \times s \\ \vdots \\ A(s)n_{is} \times s \end{bmatrix}_{n_{i.} \times s}, \tag{2.124}$$

$$U = \begin{bmatrix} A(1)n_{1.} \times r \\ \vdots \\ A(r)n_{r.} \times r \end{bmatrix}_{n_{..} \times r}, \tag{2.125}$$

$$n_{ij} = \text{the number of observations in } ij\text{th cell,} \tag{2.126}$$

$$n_{i.} = n_{i1} + \dots + n_{is}. \tag{2.127}$$

The covariance matrix of the observations is a block diagonal matrix

$$\sigma^2 H = \sigma^2 \, \mathrm{Diag}\{(I_{n1.} + \gamma_1 J), \dots , (I_{nr.} + \gamma_1 J)\} \tag{2.128}$$

and its inverse is

$$\frac{1}{\sigma^2} H^{-1} = \frac{1}{\sigma^2} \; \mathrm{Diag}\{(I - \frac{\gamma_1}{1 + n_{1.}\gamma_1} J), \dots, (I - \frac{\gamma_1}{1 + n_{r.}\gamma_1} J)\}. \tag{2.129}$$

Let us define the matrix X as follows

$$X = [1|T|U]; \tag{2.130}$$

Then it can be easily verified that the normal equations

$$X'H^{-1}X \begin{bmatrix} \hat{\mu} \\ \hat{\tau} \\ \hat{\beta} \end{bmatrix} = X'H^{-1}Y, \tag{2.131}$$

where β is a dummy null vector, may be simplified to

$$\hat{\mu}\sum_i \frac{n_{i.}}{1 + n_{i.}\gamma_1} + \sum_i \frac{\hat{\beta}_i n_{i.}}{1 + n_{i.}\gamma_1} + \sum_j \hat{\tau}_j h_j = \sum_i \frac{Y_{i..}}{1 + n_{i.}\gamma_1}, \tag{2.132}$$

$$\hat{\mu} \frac{n_{i.}}{1 + n_{i.}\gamma_1} + \hat{\beta}_i \frac{n_i}{1 + n_{i.}\gamma_1} + \sum_j \hat{\tau}_j \frac{n_{ij}}{1 + n_{i.}\gamma_1} = \frac{Y_{i..}}{1 + n_{i.}\gamma_1}, \tag{2.133}$$

$$i = 1, \dots, s,$$

$$\hat{\mu} h_j + \sum \hat{\beta}_i \frac{n_{ij}}{1 + n_{i.}\gamma_1} + \sum_k \hat{\tau}_k c_{jk} = Y_{.j.} - g_j \tag{2.134}$$

$$j = 1, \dots, r,$$

where

$$h_j = \sum_i \frac{n_{ij}}{1 + n_{i.}\gamma_1}, \tag{2.135}$$

$$c_{jj} = n_{.j} - \sum_i \frac{n_{ij}^2 \gamma_1}{1 + n_{i.}\gamma_1}, \tag{2.136}$$

$$c_{jk} = - \sum_i \frac{n_{ij} n_{ik} \gamma_1}{1 + n_{i.}\gamma_1}, \tag{2.137}$$

$$g_j = \gamma_1 \sum \frac{n_{ij} Y_{i..}}{1 + n_{i.}\gamma_1}, \tag{2.138}$$

$$Y_{i..} = \text{total of observations in block } i, \tag{2.139}$$

$$Y_{.j.} = \text{total of observations that received treatment } j. \tag{2.140}$$

Now we will specialize the two-way classification with unequal numbers per subclass to balanced incomplete block design with the following parameters:

v is the number of treatments,

b is the number of blocks,

r is the number of blocks containing any treatment,

k is the number of plots per block,

λ is the number of times any two treatments appear together in the same block.

Now it is easy to verify that

$$h_j = \frac{r}{1 + k\gamma_1}, \tag{2.141}$$

$$c_{jj} = r \frac{1 + (k - 1)\gamma_1}{1 + k\gamma_1}, \tag{2.142}$$

$$c_{jk} = - \lambda \frac{\gamma_1}{1 + k\gamma_1}, \tag{2.143}$$

$$g_j = \frac{\gamma_1}{1 + k\gamma_1} \sum_i n_{ij} Y_{i..} \quad . \tag{2.144}$$

The normal equations (2.132), (2.133) and (2.134) are readily simplified to

$$\hat{\mu}\,\frac{bk}{1+k\gamma_1} + \sum_i \hat{\beta}_i\,\frac{k}{1+k\gamma_1} + \sum_j \tau_j\,\frac{r}{1+k\gamma_1} = \frac{Y_{..}}{1+k\gamma_1}, \tag{2.145}$$

$$\hat{\mu}\,\frac{k}{1+k\gamma_1} + \hat{\beta}_i\,\frac{k}{1+k\gamma_1} + \Sigma\hat{\tau}_j n_{ij}\,\frac{1}{1+k\gamma_1} = \frac{Y_{i.}}{1+k\gamma_1}, \tag{2.146}$$

$i = 1, \ldots, r,$

$$\hat{\mu}\,\frac{r}{1+k\gamma_1} + \sum_i \hat{\beta}_i n_{ij}\frac{1}{1+k\gamma_1} + \hat{\tau}_j\,\frac{r+\lambda v\gamma_1}{1+k\gamma_1} - \Sigma\hat{\tau}_j\,\frac{\lambda\gamma_1}{1+k\gamma_1} = Y_{.j} - g_j, \tag{2.147}$$

$j = 1, \ldots, v.$

To solve the normal equations (2.145), (2.146) and (2.147) we shall rewrite (2.146) as follows:

$$(\hat{\mu} + \beta_i) = \frac{Y_{i.}}{k} - \frac{1}{k}\sum_j \hat{\tau}_j n_{ij.}, \tag{2.148}$$

and (2.147) in the following form

$$\sum_i n_{ij}\;(\hat{\mu} + \hat{\beta}_i) = (1+k\gamma_1)(Y_{.j} - g_j) - \frac{r + v\gamma_1}{1+k\gamma_1}\hat{\tau}_j + \Sigma\tau_j\,\lambda\gamma_1. \tag{2.149}$$

From (2.148)

$$\sum_i n_{ij}\;(\hat{\mu} + \hat{\beta}_i) = \frac{1}{k}\Sigma n_{ij}Y_{i.} - \frac{1}{k}(r-\lambda)\hat{\tau}_j\,. \tag{2.150}$$

Setting

$$\sum_j \tau_j = 0 \tag{2.151}$$

and equating the right hand sides of equations (2.149) and (2.150) we obtain

$$\hat{\tau}_j = \frac{k}{\lambda v}(Y_{.j} - \frac{1}{k}\sum_i n_{ij} Y_{i.}). \tag{2.152}$$

Substituting (2.152) in (2.148) we get

$$\hat{\mu} + \hat{\beta}_i = \frac{Y_{i.}}{k} - \frac{1}{\lambda v}\sum_j n_{ij} \quad (Y_{.j} - \frac{1}{k}\sum_i n_{ij} Y_{i.}). \tag{2.153}$$

From (2.145), (2.152) and (2.153) we have

$$\hat{\mu} = \frac{Y_{..}}{bk}, \tag{2.154}$$

and

$$\hat{\beta}_i = \frac{Y_{i.}}{k} - \frac{Y_{..}}{bk} - \frac{1}{\lambda v}\sum_j n_{ij}(Y_{.j} - \frac{1}{k}\sum_i n_{ij} Y_{i.}). \tag{2.155}$$

Using the model (2.122) we can prove easily that

$$\mathrm{var}(\hat{\tau}_j) = \frac{k(v-1)}{\lambda v^2}\sigma^2, \tag{2.156}$$

$$\mathrm{cov}(\hat{\tau}_j, \hat{\tau}_k) = -\frac{k}{\lambda v^2}\sigma^2, \tag{2.157}$$

$$\mathrm{var}(\hat{\beta}_i) = \sigma^2 \frac{(b-1)}{bk}(1 + k\gamma_1), \tag{2.158}$$

$$\mathrm{cov}(\hat{\beta}_i, \hat{\beta}_j) = -\frac{\sigma^2}{bk}(1 + k\gamma_1), \tag{2.159}$$

$$\mathrm{cov}(\hat{\mu}, \hat{\tau}_i) = 0, \tag{2.160}$$

$$\mathrm{cov}(\hat{\mu}, \hat{\beta}_i) = 0, \tag{2.161}$$

$$\mathrm{cov}(\tau_j, \hat{\beta}_i) = 0. \tag{2.162}$$

It is easily seen that

$$\mathrm{Reg} = \frac{Y_{..}^2}{vk(1 + k\gamma_1)} + \frac{k}{\lambda v}\Sigma Y_{.j}^2 - \frac{1}{\lambda v}[1 + \frac{(k+1)\gamma_1}{1 + k\gamma_1}] \sum_{ij} n_{ij} Y_{.j} Yi.$$

$$+ \frac{1}{1 + k\gamma_1} (1 + \frac{k - \lambda}{\lambda v}) \Sigma Y_{i.}^2 /k + \frac{\gamma_1}{\lambda v(1 + k\gamma_1)} \sum_j (\sum_i n_{ij} Y_{i.}/k)^2, \quad (2.163)$$

$$Y'H^{-1}Y = Y'Y' - \frac{\gamma_1}{1 + k\gamma_1} \Sigma Y_{i.}^2 , \quad (2.164)$$

$$\text{Res} = Y'H^{-1}Y - \text{Reg.} \quad (2.165)$$

From equations (2.158) and (2.159) we can write the covariance matrix of $\hat{\beta}$ as follows .

$$\Sigma(\gamma_1) = \frac{\sigma^2}{k} (1 + k\gamma_1)[I - \frac{1}{b} J]. \quad (2.166)$$

It is readily seen that a confidence interval may be based on the function

$$Z(\gamma_1) = \frac{k(bk - b - v + 1)\sum_i \hat{\beta}_i^2}{(1 + k\gamma_1)(b - 1) \text{ Res}} \quad . \quad (2.167)$$

After simple algebraic manipulations we obtain the confidence interval

$$\frac{k(bk - b - v + 1)\Sigma\hat{\beta}_i^2}{F_2 k(b - 1)B} - \frac{A}{kB} \le \gamma_1 \le \frac{k(bk - b - v + 1)\Sigma\hat{\beta}_i^2}{F_1 k(b - 1)B} - \frac{A}{kB} , \quad (2.168)$$

where

$$A = Y'Y - \frac{k}{\lambda v} \Sigma Y_{.j}^2 - \frac{1}{\lambda v} \Sigma n_{ij} Y_{i.} Y_{.j} - (1 + \frac{k - \lambda}{\lambda v}) \Sigma Y_{i.}^2 /k - Y_{..}^2/vk, \quad (2.169)$$

$$B = Y'Y - \frac{k}{\lambda v} \; Y_{.j}^2 + \frac{1}{k\,v} (2K + 1) \;\; n_{ij} Y_{i.} Y_{j.}$$

$$- \frac{k - \lambda}{k\lambda v} \Sigma Y_{i.}^2 /k - \frac{1}{k\lambda v} \sum_j (\sum_{i.} n_{ij} Y_{i.}/k)^2. \quad (2.170)$$

The least squares estimates (2.152), (2.154) and (2.155) are identical to the simple least squares estimates; thus the covariance matrix of $\hat{\beta}$ is given by (2.166). We can derive Wald's confidence interval from the function

$$Z_1(\gamma_1) = \frac{k(bk - b - v + 1)\Sigma\hat{\beta}_i^2}{(1 + k\gamma_1)(b - 1)\ ESSq}, \tag{2.171}$$

where

$$ESSq = Y'Y - \Sigma Y_{i.}^2/k - \frac{k}{\lambda v}\Sigma(Y._j - \frac{1}{k}\Sigma n_{ij}Y_{i.})^2, \tag{2.172}$$

is the Error Sum of Squares assuming fixed effect model. It can be easily seen that Wald's confidence interval is given by the inequality

$$\frac{1}{k}[\frac{k(bk - b - v + 1)\Sigma\hat{\beta}_i^2}{F_2(b - 1)\ ESSq} - 1] \leq \gamma_1 \leq \frac{1}{k}[\frac{k(bk - b - v + 1)\Sigma\hat{\beta}_i^2}{F_1(b - 1)ESSq} - 1]. \tag{2.173}$$

CHAPTER III

UNBALANCED DESIGNS

In this chapter we shall consider two unbalanced designs. First we shall construct a confidence interval for γ_1 in one-way classification with unequal numbers per sub class. We shall see that Hartley and Rao's [10] confidence region is identical to a confidence interval derived by Wald [20]. Then we shall consider an unbalanced nested classification with unequal levels of the secondary factor and unequal numbers of observations per secondary factor. Three confidence regions for γ_1 and γ_2 will be derived based on the procedure outlined in Hartley and Rao [10]. Wald's [22] confidence region for γ_1 and γ_2 will be derived by two methods then a confidence region for γ_2 will be derived via Wald's [22] procedure.

3.1 One-way Classification

The general one-way classification model with unequal numbers per subclass is

$$Y = 1\mu + U_1 b_1 + e, \tag{3.1}$$

where

$$U_1 = \begin{bmatrix} A(1)_{n_1 x r} \\ \vdots \\ A(r)_{n_r x r} \end{bmatrix}_{n. x r}, \tag{3.2}$$

$$n_i = \text{the number of observations in the ith class,} \tag{3.3}$$

$$n_. = n_1 + \ldots + n_r \,, \tag{3.4}$$

and the matrices A(k) were defined in chapter II.

Clearly the covariance matrix of the observations is a block diagonal matrix

$$\sigma^2 H = \sigma^2 \text{ Diag } \{(I_{n_1} + \gamma_1 J), \ \ldots, \ (I_{n_r} + \gamma_1 J)\}, \tag{3.5}$$

and

$$H^{-1} = \text{Diag } \{(I - \frac{\gamma_1}{1+n_1\gamma_1} J), \ \ldots, \ (I - \frac{\gamma_1}{1+n_r\gamma_1} J)\}. \tag{3.6}$$

To derive a confidence region for γ_1 via Hartley and Rao's [10] method we will choose W as a base matrix, where

$$W = [1 | U_1^*], \tag{3.7}$$

$$U_1^* = \begin{bmatrix} A(1)_{n_1 x r-1} \\ \cdot \\ \cdot \\ \cdot \\ A(r-1)_{n_{r-1} x r-1} \\ -\delta_r^{-1} \, 1 \, \Delta' \end{bmatrix}_{n_. x r-1} \tag{3.8}$$

$$\Delta' = [\delta_1 \ , \ \ldots \ , \ \delta_{r-1}] \ , \tag{3.9}$$

$$\delta_i = \frac{n_i}{1+n_i\gamma_1} \ , \ i = 1, \ \ldots \ , \ r \tag{3.10}$$

The normal equations for the least squares estimates of μ and the dummy null vector β are

$$W'H^{-1}W\begin{bmatrix}\hat{\mu}\\ \hat{\beta}\end{bmatrix} = W'H^{-1}Y\ . \tag{3.11}$$

Simplifying equation (3.11) we get

$$\begin{bmatrix}\sum \delta_i & 0\\ 0 & \mathrm{Diag}\{\delta_1,\ldots,\delta_{r-1}\}+\delta_r^{-1}\Delta\Delta'\end{bmatrix}\begin{bmatrix}\hat{\mu}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\sum \delta_i\, Y_{i.}/n_i\\ \delta_1(Y_{1.}/n_1 - Y_{r.}/nr)\\ \vdots\\ \delta_{r-1}(Y_{r-1.}/n_{r-1} - Y_{r.}/n_r)\end{bmatrix}. \tag{3.12}$$

where

$Y_{i.}$ = the total of the observations in the ith subclass.

It is trivial to verify that

$$[W'H^{-1}W]^{-1} = \begin{bmatrix}(\sum \delta_i)^{-1} & 0\\ 0 & \mathrm{Diag}\{\delta_1^{-1},\ \ldots\ ,\ \delta_{r-1}^{-1}\} - (\sum\delta_i)^{-1}\ J\end{bmatrix}. \tag{3.13}$$

After algebraic simplification we can put the solution of (3.12) in the following form:

$$\begin{bmatrix}\hat{\mu}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\dfrac{\sum \delta_i\, Y_{i.}/n_i}{\sum \delta_i}\\ Y_{1.}/n_1 - \dfrac{\sum\delta_i\, Y_{i.}/n_i}{\sum\delta_i}\\ \vdots\\ Y_{r-1.}/n_{r-1} - \dfrac{\sum\delta_i\, Y_{i.}/n_i}{\sum\delta_i}\end{bmatrix}. \tag{3.14}$$

The quantities Reg and Res are given by the following equations

$$\text{Reg} = \frac{[\sum\delta_i\, Y_{i.}/n_i]^2}{\sum\delta_i} + [\sum\delta_i(Y_{i.}/n_i)^2 - \frac{[\sum\delta_i\, Y_{i.}/n_i]^2}{\sum\delta_i}], \tag{3.15}$$

$$\text{Res} = Y'Y - \sum Y^2_{i.}/n_i \ . \tag{3.16}$$

Since the covariance matrix of (3.14) is given by

$$\sigma^2[W'H^{-1}W]^{-1} \ ,$$

it can be readily seen that $\hat{\mu}$ and $\hat{\beta}$ are independent. Plainly

$$\textstyle\sum_{22.1}(\gamma_1) = \text{Diag}\{\delta_1^{-1}, \ \ldots \ , \ \delta_{r-1}^{-1}\} - (\sum\delta_i)^{-1}J, \tag{3.17}$$

and

$$\textstyle\sum_{22.1}^{-1}(\gamma_1) = \text{Diag}\{\delta_1, \ \ldots \ , \ \delta_{r-1}\} + \delta_r^{-1}\Delta\Delta' \ . \tag{3.18}$$

The confidence region is based on the function

$$Z(\gamma_1) = \frac{\sum(n_i-1)\ \hat{\beta}'\ \sum_{22.1}^{-1}\ (\gamma_1)\hat{\beta}}{(r-1)\ \text{Res}} \ . \tag{3.19}$$

After simplifications (3.19) becomes

$$Z(\gamma_1) = \frac{\sum(n_i-1)\sum\delta_i(Y_{i.}/n_i - \hat{\mu})^2}{(r-1)\ \text{Res}} \ . \tag{3.20}$$

The function (3.20) was derived by Wald [20]. He showed that $Z(\gamma_1)$ is a monotonic function of γ_1. Thus in this case the confidence region for γ_1 is a confidence interval which agrees with Wald's confidence interval.

3.2 Nested Classification

The unbalanced nested classification random model is specified by the general mathematical model

$$Y = 1\mu + U_1 b_1 + U_2 b_2 + e, \tag{3.21}$$

where

$$U_1 = \begin{bmatrix} A(1)_{n_{1.} x r} \\ \vdots \\ A(r)_{n_{r.} x r} \end{bmatrix}_{n_{..} x r} \tag{3.22}$$

$$U_2 = \mathrm{Diag}\,\{B_1, \ldots, B_r\}, \tag{3.23}$$

$$B_i = \begin{bmatrix} A(1)_{n_{i_1} x s_i} \\ \vdots \\ A(s_i)_{n_{is_i} x s_i} \end{bmatrix}_{n_{i.} x s_i}, \tag{3.24}$$

r = the number of levels of the primary factor,

s_i = the number of levels of the secondary factor within the ith primary factor,

n_{ij} = the number of observations in the jth secondary level within the ith primary level,

$n_{i.} = n_{i_1} + \ldots + n_{is_i}$,

$n_{..} = n_{i.} + \ldots + n_{r.}$,

and the matrices A(k) were defined previously.

Clearly the covariance matrix of the observations is

$$\sigma^2 H = \sigma^2[I + \gamma_1 U_1U_1' + \gamma_2 U_2U_2'], \tag{3.25}$$

where

$$U_1U_1' = \text{Diag }\{J_{n_{1.}}, \ldots, J_{n_{r.}}\}, \tag{3.26}$$

$$U_2U_2' = \text{Diag }\{J_{n_{11}}, \ldots, J_{n_{rs_r}}\}. \tag{3.27}$$

It is convenient to define

$$C_i = \left[\begin{array}{ccc} I_{n_{i1}} + (\gamma_1 + \gamma_2)J, & \ldots, & \gamma_1 J \\ \hline \gamma_1 J & , \ldots, & I_{n_{is_i}} + (\gamma_1 + \gamma_2)J \end{array}\right]. \tag{3.28}$$

Now it is trivial to verify that

$$H = \text{Diag }\{C_1, \ldots, C_r\}. \tag{3.29}$$

Following Rao [14] it can be easily verified that

$$C_i^{-1} = \left[\begin{array}{ccc} I_{n_{i1}} + \delta_{11}^i J, & \ldots, & \delta_{1s_i}^i J \\ \hline \delta_{s_i1}^i J, & , \ldots, & I_{n_{is_i}} + \delta_{s_is_i}^i J \end{array}\right], \tag{3.30}$$

where δ_{jk}^i is the jkth element of the matrix

$$\Delta^i = -\gamma_2[\text{Diag}\{\lambda_{i1}, \ldots, \lambda_{is_i}\} + \frac{\gamma_1}{\gamma_2(1+\gamma_1 \sum_j n_{ij}\lambda_{ij})} \Lambda_i\Lambda_i'], \tag{3.31}$$

where

$$\Lambda_i' = [\lambda_{i1}, \ldots, \lambda_{is_i}], \tag{3.32}$$

$$\lambda_{ij} = \frac{1}{1 + n_{ij}\gamma_2}. \tag{3.33}$$

Since H is a block diagonal matrix, plainly

$$H^{-1} = \text{Diag }\{C_1^{-1}, \ldots, C_r^{-1}\}. \tag{3.34}$$

In order to apply the procedure outlined by Hartley and Rao [10] we will choose a base matrix W defined as follows:

$$W = [1|U_1^*|U_2^*], \tag{3.35}$$

where

$$U_1^* = \begin{bmatrix} A(1)_{n_{1.}xr-1} \\ \vdots \\ A(r-1)_{n_{r-1.}xr-1} \\ -\xi_r^{-1}\ 1\ \eta', \end{bmatrix}_{n_{..}xr-1} \tag{3.36}$$

$$\eta' = [\xi_1, \ldots , \xi_{r-1}], \tag{3.37}$$

$$\xi_i = \sum_j n_{ij}\ \lambda_{ij}\ (1 + \gamma_1 \sum_j n_{ij}\ \lambda_{ij})^{-1}, \tag{3.38}$$

$$i = 1, \ldots , r ,$$

$$U_2^* = \mathrm{Diag}\ \{B_1^* , \ldots , B_r^*\}, \tag{3.39}$$

$$B_i^* = \begin{bmatrix} A(1)_{n_{i1}xs_i-1} \\ \vdots \\ A(s_i-1)_{n_{is_i-1}xs_i-1} \\ -\omega_{is_i}^{-1}\ 1\ \Psi_i' \end{bmatrix}_{n_{i.}xs_i-1}, \tag{3.40}$$

$$\Psi_i' = [\omega_{i1}, \ldots , \omega_{is_i-1}], \tag{3.41}$$

$$\omega_{ij} = n_{ij}\ \lambda_{ij} , \tag{3.42}$$

$$i = 1, \ldots, r,$$

$$j = 1, \ldots, s_i .$$

A consequence of such choice of W is

$$U_1^{*\prime} H^{-1} 1 = 0 , \tag{3.43}$$

$$U_2^{*\prime} H^{-1} 1 = 0 , \tag{3.44}$$

$$U_1^{*\prime} H^{-1} U_2^{*} = 0. \tag{3.45}$$

As usual we denote by β a dummy null vector.

We divide β into $(r-1)$ and $\sum(s_i-1)$ subvectors as follows

$$\beta = \begin{bmatrix} \beta_1 \\ \\ \beta_2 \end{bmatrix} . \tag{3.46}$$

The normal equations for the least squares estimates of μ, β_1 and β_2 are

$$\begin{bmatrix} 1' H^{-1} 1 & 0 & 0 \\ 0 & U_1^{*\prime} H^{-1} U_1^{*} & 0 \\ 0 & 0 & U_2^{*\prime} H^{-1} U_2^{*} \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = W' H^{-1} Y. \tag{3.47}$$

It is trivial to verify that

$$1' H^{-1} 1 = \sum \xi_i ,$$

$$U_1^{*\prime} H^{-1} U_1^{*} = \mathrm{Diag}\, \{\xi_1, \ldots, \xi_{r-1}\} + \xi_r^{-1} \eta\eta' , \tag{3.48}$$

$$U_2^{*\prime} H^{-1} U_2^{*} = \mathrm{Diag}\, \{B_1^{*\prime} C_1^{-1} B_1^{*} , \ldots , B_r^{*\prime} C_r^{-1} B_r^{*}\}, \tag{3.49}$$

$$B_i^{*\prime} C_i^{-1} B_i^{*} = \mathrm{Diag}\{\omega_{i1}, \ldots, \omega_{is_i-1}\} + \omega_{is_i}^{-1} \Psi_i \Psi_i' . \tag{3.50}$$

To obtain $W'H^{-1}Y$ it is convenient to write

$$C_i^{-1} = I + \begin{bmatrix} \delta^i_{11} J, \ldots, \delta^i_{1s_i} J \\ \hline \delta^i_{s_i1} J, \ldots, \delta^i_{s_is_i} J \end{bmatrix}, \qquad (3.51)$$

$$B_i \Delta^i B_i' = \begin{bmatrix} \delta^i_{11} J, \ldots, \delta^i_{1s_i} J \\ \hline \delta^i_{s_i1} J, \ldots, \delta^i_{s_is_i} J \end{bmatrix}, \qquad (3.52)$$

$$Y' = [Y_1', \ldots, Y_r'], \qquad (3.53)$$

where B_i is defined by (3.24) and Y_i is the sub vector of the observations in the ith primary level.

After simple algebraic manipulations it can be easily verified that

$$C_i^{-1} Y_i = Y_i - \gamma_2 B_i \begin{bmatrix} \omega_{i1} (Y_{i1.}/n_{i1} + \tilde{Y}_{i..}/n_{i1}) \\ \vdots \\ \omega_{is_i} (Y_{is_i.}/n_{is_i} + \tilde{Y}_{i..}/n_{is_i}) \end{bmatrix}, \qquad (3.54)$$

where

$$\tilde{Y}_{i..} = \frac{\gamma_1 \sum\limits_j (\frac{\omega_{ij}}{n_{ij}}) Y_{ij.}}{\gamma_2 (1 + \gamma_1 \sum\limits_j \omega_{ij})}, \qquad (3.55)$$

$Y_{ij.}$ = the total of the observations in the jth secondary level within the ith primary level.

Finally let us define

$$X_i = 1'C_i^{-1} Y_i, \ (i = 1, \ldots, r). \tag{3.56}$$

It is easily seen that

$$X_i = \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}}{1 + \gamma_1 \sum_j \omega_{ij}} . \tag{3.57}$$

We can write the right hand side of the normal equations (3.47) as follows:

$$W'H^{-1}Y = \begin{bmatrix} 1' H^{-1}Y \\ U_1^{*\prime} H^{-1}Y \\ U_2^{*\prime} H^{-1} Y \end{bmatrix} . \tag{3.58}$$

It is trivial that

$$1' H^{-1}Y = \sum_i \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}}{1 + \gamma_1 \sum_j \omega_{ij}} . \tag{3.59}$$

$$U_1{*}'H^{-1}Y = \begin{bmatrix} X_1 \\ \vdots \\ X_{r-1} \end{bmatrix} \tag{3.60}$$

$$U_2^{*\prime}H^{-1}Y = \begin{bmatrix} B_1^{*\prime}C_1^{-1}Y \\ \vdots \\ B_r^{*\prime}C_r^{-1}Y \end{bmatrix} \tag{3.61}$$

where

$$X_i = (1 + \gamma_1 \sum_j \omega_{ij})^{-1} \sum_j \omega_{ij} (Y_{ij.}/n_{ij.} - \frac{\sum_j \frac{\omega_{rj}}{n_{rj}} Y_{rj.}}{\sum_j \omega_{rj}}), \tag{3.62}$$

$$B_i^{*\prime} C_i^{-1} Y = \begin{bmatrix} \frac{\omega_{i1}}{n_{i1}} (Y_{i1}./n_{i1} - Y_{is_i}./n_{is_i}) \\ \vdots \\ \frac{\omega_{is_i}}{n_{is_i}} (Y_{is_i}./n_{is_i} - Y_{is_i}./n_{is_i}) \end{bmatrix} . \tag{3.63}$$

To find the solution of the normal equations we need to find

$$(W'H^{-1}W)^{-1} = \begin{bmatrix} (1'H^{-1}1)^{-1} & 0 & 0 \\ 0 & (U_1^{*\prime}H^{-1}U_1^*)^{-1} & 0 \\ 0 & 0 & (U_2^{*\prime}H^{-1}U_2^*)^{-1} \end{bmatrix} . \tag{3.64}$$

It may be verified by direct multiplication that

$$(U_1^* H^{-1} U_1^*)^{-1} = \text{Diag}\{\xi^{-1}, \ldots, \xi_{r-1}^{-1}\} - (\Sigma\xi_i)^{-1} J, \tag{3.65}$$

$$(U_2^{*\prime}H^{-1}U_2^*)^{-1} = \text{Diag}\{(B_1^{*\prime}C_1^{-1}B_1^*)^{-1}, \ldots , (B_r^{*\prime}C_r^{-1}B_r^*)^{-1}\} \tag{3.66}$$

where

$$(B_i^{*\prime}C_i^{-1}B_i^*)^{-1} = \text{Diag}\{\omega_{i_1}^{-1}, \ldots , \omega_{is_i-1}^{-1}\} - (\sum_j \omega_{ij})^{-1} J. \tag{3.67}$$

Now it can be easily verified that the solution of the normal equations is

$$\begin{bmatrix} \hat{\mu} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mu} \\ \hat{a}_1 - \hat{\mu} \\ \vdots \\ \hat{a}_{r-1} - \hat{\mu} \\ Y_{11.}/n_{11} - \hat{a}_1 \\ \vdots \\ Y_{1s_1-1.}/n_{1s_1-1} - \hat{a}_1 \\ \vdots \\ Y_{r1.}/n_{r1} - \hat{a}_r \\ \vdots \\ Y_{rs_r-1.}/n_{rs_r-1} - \hat{a}_r \end{bmatrix}, \tag{3.68}$$

where

$$\hat{\mu} = \frac{\sum_i \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}}{1 + \gamma_1 \sum_j \omega_{ij}}}{\sum_i \frac{\sum \omega_{ij}}{1 + \gamma_1 \sum \omega_{ij}}}, \tag{3.69}$$

$$\hat{a}_i = \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij\cdot}}{\sum_j \omega_{ij}} \quad . \tag{3.70}$$

After considerable algebraic manipulations we can write the quantities Reg and Res as follows:

$$\text{Reg} = \frac{\left[\sum_i \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij\cdot}}{1 + \gamma_1 \sum_j \omega_{ij}}\right]^2}{\sum_i \frac{\sum_j \omega_{ij}}{1 + \gamma_1 \sum_j \omega_{ij}}} + \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{(1 + \gamma_1 \sum_j \omega_{ij})(\sum_j \omega_{ij})} .$$

$$- \frac{\left[\sum_i \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij\cdot}}{1 + \gamma_1 \sum_j \omega_{ij}}\right]^2}{\sum_i \frac{\sum_j \omega_{ij}}{1 + \gamma_1 \sum_j \omega_{ij}}} + \sum_{i=1}^{r} \sum_{j=1}^{s_i} \omega_{ij}(Y_{ij.}/n_{ij})^2$$

$$- \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{\sum_j \omega_{ij}} \quad , \tag{3.71}$$

$$\text{Res} = Y'Y - \sum Y_{ij\cdot}^2/n_{ij} \quad . \tag{3.71}$$

We partition the covariance matrix of the estimates (3.68) as follows:

$$\Sigma(\gamma) = \begin{bmatrix} \Sigma_{11}(\gamma) & 0 & 0 \\ 0 & \Sigma_{22}(\gamma) & 0 \\ 0 & 0 & \Sigma_{33}(\gamma) \end{bmatrix} , \tag{3.72}$$

where

$$\Sigma_{11}(\gamma) = (\sum_i \xi_i)^{-1} \tag{3.73}$$

$$\Sigma_{22}(\gamma) = (U_1^{*\prime} H^{-1} U_1^*)^{-1} \tag{3.74}$$

$$\Sigma_{33}(\gamma) = (U_2^{*\prime} H^{-1} U_2^*)^{-1}. \tag{3.75}$$

Let us note first that the estimates $\hat{\mu}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent. This is obvious from (3.72). Thus

$$\Sigma_{22.13}(\gamma) = \Sigma_{22}(\gamma) , \tag{3.76}$$

$$\Sigma_{33.12}(\gamma) = \Sigma_{33}(\gamma) , \tag{3.77}$$

and if we let

$$\bar{\Sigma}(\gamma) = \begin{bmatrix} \Sigma_{22}(\gamma) & 0 \\ 0 & \Sigma_{33}(\gamma) \end{bmatrix} , \tag{3.78}$$

then

$$\bar{\Sigma}_{\cdot 1}(\gamma) = \bar{\Sigma}(\gamma). \tag{3.79}$$

Following Hartley and Rao [10] we construct the following confidence regions for γ_1 and γ_2:

(1) A confidence region for γ_1 and γ_2 may be based on the function

$$Z_1(\gamma_1,\gamma_2) = \frac{(n_{..} - \sum s_i)(\hat{\beta}_1' \Sigma_{22}^{-1}(\gamma)\hat{\beta}_1 + \hat{\beta}_2' \Sigma_{33}^{-1}(\gamma)\hat{\beta}_2)}{(\sum s_i - 1)\ \text{Res}} \tag{3.80}$$

It may be verified that

$$Z_1(\gamma_1,\gamma_2) = \frac{(n_{..} - \sum s_i)\ C}{(\sum s_i - 1)\ \text{Res}}\ , \tag{3.81}$$

where

$$C = \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{(1 + \gamma_1 \sum_j \omega_{ij})(\sum_j \omega_{ij})} - \frac{\left[\sum_i \frac{\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}}{1 + \gamma_1 \sum_j \omega_{ij}}\right]^2}{\sum_i \frac{\sum_j \omega_{ij}}{1 + \gamma_1 \sum_j \omega_{ij}}}$$

$$+ \sum_i \sum_j \omega_{ij}(Y_{ij.}/n_{ij})^2 - \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{\sum_j \omega_{ij}}\ . \tag{3.82}$$

(2) We can also use the following functions to construct confidence region on γ_1 and γ_2:

$$Z_2(\gamma_1,\gamma_2) = \frac{(n_{..} - \sum s_i)\hat{\beta}_1' \Sigma_{22}^{-1}(\gamma)\hat{\beta}_1}{(r-1)\ \text{Res}}\ , \tag{3.83}$$

A confidence region for γ_2 may be based upon

$$Z_3(\gamma_2) = \frac{(n_{..} - \sum s_i)\hat{\beta}_2' \Sigma_{33}^{-1}(\gamma)\hat{\beta}_2}{(\sum s_i - r)\ \text{Res}}\ . \tag{3.84}$$

It may be easily verified that the functions $Z_2(\gamma_1,\gamma_2)$ and $Z_3(\gamma_2)$ simplify to

$$Z_2(\gamma_1,\gamma_2) = \frac{(n_{..} - \sum s_i)\ A}{(r-1)\ \text{Res}}\ , \tag{3.85}$$

$$Z_3\ (\gamma_2) = \frac{(n_{..} - \sum s_i)\ B}{(r-1)\ \text{Res}}\ , \tag{3.86}$$

where

$$A = \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{(1 + \gamma_1 \sum_j \omega_{ij})(\Sigma\omega_{ij})} - \frac{\left[\sum_i \frac{\sum \frac{\omega_{ij}}{n_{ij}} Y_{ij.}}{1 + \gamma_1\ \Sigma\omega i_j}\right]^2}{\sum_i \sum \frac{\omega_{ij}}{1 + \gamma_1\ \Sigma\omega_{ij}}}\ , \tag{3.87}$$

$$B = \sum_i \sum_j \omega_{ij}(Y_{ij.}/n_{ij})^2 - \sum_i \frac{\left[\sum_j \frac{\omega_{ij}}{n_{ij}} Y_{ij.}\right]^2}{\sum_j \omega_{ij}}\ . \tag{3.88}$$

(3) Finally, we can construct a confidence region based on the function

$$Z_4(\gamma_1,\gamma_2) = \frac{(\sum s_i - r)\ \hat{\beta}_1'\ \sum_{22}^{-1}\ (\gamma)\ \hat{\beta}_1}{(r-1)\hat{\beta}_2'\ \sum_{33}^{-1}\ (\gamma)\hat{\beta}_2}\ , \tag{3.89}$$

or its inverse

$$Z_5(\gamma_1,\gamma_2) = [Z_4(\gamma_1,\gamma_2)]^{-1}. \tag{3.89}$$

We will show that $Z_4(\gamma_1,\gamma_2)$ is distributed like F distribution. Since $\sum_{22}(\gamma)$ is the covariance matrix of $\hat{\beta}_1$, it follows immediately that the quadratic form

$$q_1 = \hat{\beta}_1'\ \sum_{22}^{-1}\ (\gamma)\ \hat{\beta}_1 \tag{3.90}$$

follows the chi-square distribution with (r-1) degrees of freedom. Similarly, the quadratic form

$$q_2 = \hat{\beta}'_2, \textstyle\sum_{33}^{-1} (\gamma) \hat{\beta}_2 \tag{3.91}$$

follows the chi-square distribution wit $(\sum s_i - r)$ degrees of freedom. The quadratic forms q_1 and q_2 are independent since $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent. Thus $Z_4(\gamma_1,\gamma_2)$ follows the F distribution with $(r - 1)$ and $(\sum s_i - r)$ degrees of freedom.

Thus either

$$Z_4(\gamma_1,\gamma_2) \leq F\ (\alpha;\ r - 1,\ \textstyle\sum s_i - r), \tag{3.92}$$

or

$$Z_5(\gamma_1,\gamma_2) \leq F\ (\alpha;\ \textstyle\sum s_i - r,\ r - 1), \tag{3.93}$$

defines a confidence region for γ_1 and γ_2.

We now turn to Wald's [22] confidence region. We assume that the model (3.21) is a fixed effect model. To find simple least squares estimate of b_1 and b_2 we need to reparametarize the model as follows:

$$Y = 1\mu + U_1^* b_1^* + U_2^* b_2^* + e, \tag{3.94}$$

where

$$U_1^* = \begin{bmatrix} A(1) \\ \cdot \\ \cdot \\ \cdot \\ A(r-1) \\ -n_{r.}^{-1}\ 1D' \end{bmatrix}_{n_{..} \times r-1}, \tag{3.95}$$

$$U_2^* = \text{Diag}\ \{B_1^*\ ,\ \ldots\ ,\ B_r^*\}\ , \tag{3.96}$$

$$B_i^* = \begin{bmatrix} A(1) \\ \cdot \\ \cdot \\ \cdot \\ A(r-1) \\ -n_{r.}^{-1}\, 1\, D' \end{bmatrix}_{n_{..} x r-1} \tag{3.95}$$

$$U_2^* = \text{Diag}\,\{B_1^*\ , \ \ldots\ ,\ B_r^*\}\quad , \tag{3.96}$$

$$B_i^* = \begin{bmatrix} A(1) \\ \cdot \\ \cdot \\ \cdot \\ A(s_i-1) \\ -n_{is_i}^{-1}\, 1\, D_i' \end{bmatrix}_{n_{i.} x s_i-1}\ , \tag{3.97}$$

$$D' = [n_{1.}\ , \ \ldots\ ,\ n_{r-1.}], \tag{3.98}$$

$$D_i' = [n_{i_1}\ , \ \ldots\ ,\ n_{is_i-1}], \tag{3.99}$$

and the matrices A(k) were defined previously.

Let $\hat{b}_1$ and $\hat{b}_2$ be the simple least squares estimates of b_1^* and b_2^* respectively. It can be easily verified that the normal equations which $\hat{b}_1$ and $\hat{b}_2$ are their solution are

$$\begin{bmatrix} n_{..} & 0 & 0 \\ 0 & U_1^{*\prime}U_1^* & 0 \\ 0 & & U_2^{*\prime}U_2^* \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}$$

$$= \begin{bmatrix} Y_{...} \\ n_{1.}(Y_{1..}/n_{1.} - Y_{r..}/n_{r.}) \\ \vdots \\ n_{r-1}(Y_{r-1..}/n_{r-1} - Y_{r..}/n_{r.}) \\ n_{11}(Y_{11.}/n_{11} - Y_{1s_1.}/n_{1s_1}) \\ \vdots \\ n_{1s_1-1}(Y_{1s_1-1.}/n_{1s_1} - Y_{1s_1.}/n_{1s_1}) \\ \vdots \\ n_{r1}(Y_{r1.}/n_{r1} - Y_{rs_r.}/n_{rs_r}) \\ \vdots \\ n_{rs_r-1}(Y_{rs_r-1.}/n_{rs_r-1} - Y_{rs_r.}/n_{rs_r}) \end{bmatrix}, \tag{3.100}$$

where

$$n_{..} = \sum_i \sum_j n_{ij} \tag{3.101}$$

$$U_1^{*\prime}\, U_1^* = \mathrm{Diag}\,\{n_{1.}\ , \ldots , n_{r-1.}\} + n_{r.}^{-1}\, DD', \tag{3.102}$$

$$U_2^{*\prime}\, U_2^* = \mathrm{Diag}\,\{B_1^{*\prime}\, B_1^*\ , \ldots , B_r^{*\prime}\, B_r^*\} \quad , \tag{3.103}$$

$$B_i^{*\prime}\, B_i^* = \mathrm{Diag}\,\{n_{i1}\ , \ldots , n_{is_i}^{-1}\} + n_{is_i}^{-1}\, D_iD_i' \ . \tag{3.104}$$

It may be verified by direct multiplication that

$$\begin{bmatrix} n_{..} & 0 & 0 \\ 0 & U_1^{*\prime} U_1^* & 0 \\ 0 & 0 & U_2^{*\prime} U_2^* \end{bmatrix}^{-1} = \begin{bmatrix} n_{..}^{-1} & 0 & 0 \\ 0 & (U_1^{*\prime}U_1^*)^{-1} & 0 \\ 0 & 0 & (U_2^{*\prime}U_2^*)^{-1} \end{bmatrix}, \qquad (3.105)$$

where

$$(U_1^{*\prime}\ U_1^*)^{-1} = \text{Diag}\ \{n_{1.}^{-1}\ , \ldots , n_{r-1.}^{-1}\} - n_{..}^{-1}\ J, \qquad (3.106)$$

$$(U_2^{*\prime}\ U_2^*)^{-1} = \text{Diag}\ \{(B_1^{*\prime}\ B_1^*)^{-1}, \ldots , (B_r^{*\prime}B_r^*)^{-1}\}, \qquad (3.107)$$

$$(B_i^{*\prime}\ B_i^*)^{-1} = \text{Diag}\ \{n_{i1}^{-1}, \ldots , n_{is_i-1}^{-1}\} - n_{i.}^{-1}\ J. \qquad (3.108)$$

Now it is trivial to verify that the solution of the normal equations is

$$\begin{bmatrix} \hat{\mu} \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} Y_{...}/n_{..} \\ Y_{1..}/n_{1.} - Y_{...}/n_{..} \\ \vdots \\ Y_{r-1..}/n_{r-1.} - Y_{...}/n_{..} \\ Y_{11.}/n_{11} - Y_{1..}/n_{1.} \\ \vdots \\ Y_{1s_1-1.}/n_{1s_1-1} - Y_{1..}/n_{1.} \\ \vdots \\ Y_{r1.}/n_{r1} - Y_{r..}/n_{r.} \\ \vdots \\ Y_{rs_r-1.}/n_{rs_r} - Y_{r..}/n_{r.} \end{bmatrix}, \qquad (3.109)$$

and

$$\sigma^2 \Gamma = \sigma^2 (W'W)^{-1} W'HW(W'W)^{-1} \tag{3.110}$$

is the covariance matrix of (3.109).

Following Wald [22] we can construct two confidence regions:

(1) We can construct a confidence interval for γ_2. Let us note that the covariance matrix of $\hat{b}_2$ is

$$\sigma^2 \Gamma(\hat{b}_2) = \sigma^2 (U_2^{*\prime} U_2^*)^{-1} (U_2^{*\prime} H U_2^*)(U_2^{*\prime} U_2^*)^{-1}. \tag{3.111}$$

Wald's confidence interval for γ_2 can be based on

$$Z_1(\gamma_2) = \frac{\hat{b}_2' [\Gamma(\hat{b}_2)]^{-1} \hat{b}_2}{(\sum s_i - r) \text{ EMS}} , \tag{3.112}$$

where EMS is the Error Mean Squares of the analysis of variance assuming fixed effect model. After some algebraic manipulations we can simplify (3.112) to

$$Z_1(\ _2) = \frac{\sum_i \sum_j \omega_{ij} (Y_{ij.}/n_{ij})^2 - \sum_i \frac{\left[\sum_j \omega_{ij} Y_{ij.}/n_{ij}\right]^2}{\sum_j \omega_{ij}}}{(\sum s_i - r) \text{ EMS}} . \tag{3.113}$$

In verifying that (3.112) can be reduced to (3.113) it is helpful to note that

$$(U_2^{*\prime}U^*_2)\hat{b}_2 = \begin{bmatrix} n_{11}(Y_{11.}/n_{11} - Y_{1s_1.}/n_{1s_1}) \\ \vdots \\ n_{1s_1-1}(Y_{1s_1-1.}/n_{1s_1-1} - Y_{1s_1.}/n_{1s_1}) \\ \vdots \\ n_{r1}(Y_{r1.}/n_{r1} - Y_{rs_r.}/n_{rs_r}) \\ \vdots \\ n_{rs_r-1}(Y_{rs_r-1.}/n_{rs_r-1} - Y_{rs_r.}/nrs_1) \end{bmatrix}$$

and

$$U_2^{*\prime}\ HU_2^* = \text{Diag}\ \{B_1^{*\prime}\ C_1\ B_1^*\ , \ldots , B_r^{*\prime}\ C_r\ B_r^*\} \tag{3.115}$$

$$B_i^{*\prime}\ C_r\ B_i^* = \text{Diag}\ \{n_{i_1}^2\ \omega_{i_1}^{-1}\ , \ldots , n_{is_i-1}^2\ \omega_{is_i-1}^{-1}\}$$
$$+ \omega_{is_i}^{-1}\ D_iD_i' \quad . \tag{3.116}$$

Furthermore, we note that (3.113) is identical to (3.86).

(2) We now construct a confidence region for γ_1 and γ_2. It is readily verified that the covariance matrix of $\hat{b}_1$ is

$$\sigma^2\ \Gamma(\hat{b}_1) = \sigma^2(U_1^{*\prime}U_1^*)^{-1}(U_1^{*\prime}H\ U_1^*)\ (U_1^{*\prime}\ U_1^*)^{-1}\ . \tag{3.117}$$

It is readily seen that

$$U_1^{*\prime}HU_1^* = \text{Diag}\ \{n_{1.}\rho_1\ , \ldots , n_{r-1.}\rho_{r-1}\} + n_{r.}^{-1}\rho_r DD'\ , \tag{3.118}$$

where

$$\rho_i = 1 + \gamma_1 n_{i.} + \gamma_2 \Sigma n_{ij}^2/n_{i.} \quad .$$

Wald's confidence interval is based on the function

$$Z_2(\gamma_1,\gamma_2) = \frac{\hat{b}_1' \, [\Gamma(\hat{b}_1)]^{-1} \, \hat{b}_1}{(r-1)\ \text{EMS}} \tag{3.119}$$

We can simplify (3.119) to

$$Z_2(\gamma_1,\gamma_2) = \frac{\sum_i \frac{\rho_i^{-1}}{n_{i.}} Y_{i..}^2 - \frac{(\sum \rho_i^{-1} Y_{i..})^2}{\sum \rho_i^{-1} n_{i.}}}{(r-1)\ \text{EMS}} \tag{3.120}$$

It is interesting to note that (3.120) may be derived in a different way. Let

$$\bar{Y}_{i..} = Y_{i..}/n_{i.} \quad , \tag{3.121}$$

then $\bar{Y}_{1..}, \ldots, \bar{Y}_{r..}$ are multivariate normal variables with mean 1μ and covariance

$$\sigma^2 \Delta = \sigma^2 \text{ Diag } \{ \rho_{1/n_{1.}}, \ldots, \rho_{r/n_{r.}} \} . \tag{3.122}$$

It is trivial to verify that the least squares estimate of μ is

$$\hat{\mu} = \frac{\sum_i n_i \rho_i^{-1} \bar{Y}_{i..}}{\sum n_i \rho_i^{-1}} . \tag{3.122}$$

We define the Conventional Error Mean Square to be

$$\xi = \frac{1}{\sigma^2} \sum_{i=1}^{r} n_i \rho_i^{-1} \left[\bar{Y}_{i..} - \frac{\sum_i n_i \rho_i^{-1} \bar{Y}_{i..}}{\sum n_i \rho_i^{-1}} \right]^2 . \tag{3.123}$$

It is well known that ξ is distributed like chi-square variate with r-1 degrees of freedom. We note also that ξ is the numerator of (3.120). Thus we may derive (3.120) without reference to $\Gamma(\hat{b}_1)$. This procedure follows very closely Wald's [20] approach for constructing confidence interval for γ_1 in one way classification.

Another way of constructing confidence region for γ_1 and γ_2 is by "combining" the confidence intervals (3.113) and (3.120). That is, a confidence region for γ_1 and γ_2 may be based on the function

$$Z_3(\gamma_1,\gamma_2) = \frac{[\hat{b}_1', \hat{b}_2'] \; \Gamma^{-1} \begin{bmatrix} \hat{b}_1 \\ b_2 \end{bmatrix}}{(\sum s_i - 1) \text{ EMS}} \tag{3.124}$$

This procedure requires the inversion of the following matrix

$$\begin{bmatrix} U_1^{*\prime} \; H \; U_1^* & U_1^{*\prime} \; H \; U_2^* \\ U_2^{*\prime} \; H \; U_1^* & U_2^{*\prime} \; H \; U_2^* \end{bmatrix}. \tag{3.124}$$

It is difficult to obtain such inverse because $U_1^{*\prime}$ H U_2^* is different from zero.

3.3 Summary and Conclusions

A method described by Hartley and Rao [10] is used to derive confidence region, R, for the ratios of variance components, σ_i^2, for various experimental designs. First we consider balanced designs (Chapter II). For all the cases the least squares estimates of the fixed effects and of the null vector β were identical to the simple least squares estimates. However, the confidence regions derived via Hartley and Rao's [10] method was different from the traditional analysis of variance method in the case of the Two-way Classification Model II without Interaction and the Balanced Incomplete Block Design. Such differences arise because the quantity Res defined by equation (1.11) was different from the traditional Error Sum of Squares of the analysis of variance. A necessary and sufficient condition for the equality of the quantity Res and the Error Sum of Squares of the analysis of variance was not derived.

In Chapter III we considered two unbalanced designs. For the one-way unbalanced design Hartley and Rao's [10] procedure was identical to Wald's [20] result. Our results for the unbalanced nested classification are the first confidence regions for the ratios of variances to be derived from unbalanced nested data.

It is well known that every confidence region can be translated into a test of hypothesis. In the terminology of the test of hypothesis, all the confidence regions considered in this work are "similar region" confidence regions. To study the properties of these confidence regions it might be easier to study the properties of the associated tests first.

Finally we mention that given a confidence region for a set of parameters it is possible to derive a confidence region for any function of these parameters. This fact can be used to construct a confidence region for various measures of heritability, $h^2(\gamma_i)$. Strictly speaking such a problem is one in mathematical programing.

A conservative lower confidence point for $h^2(\gamma_i)$ can be computed from the problem

$$\min_{\gamma_i} h^2(\gamma_i)$$

subject to $\gamma_i \varepsilon R$ and the upper confidence point from

$$\max_{\gamma_i} h^2(\gamma_i)$$

subject to $\gamma_i \varepsilon R$. However, since c is small the above problem can be easily solved by accepted method of numerical analysis based on scanning techniques.

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